2018

MATHEMATICS – HONOURS

Paper : CC-2

Full Marks : 65

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meaning.

1. Choose the correct alternative. Justify your answer. 2×10

(a) The value of $\sqrt[3]{i}+\sqrt[3]{-i}$ (where $\sqrt[3]{z}$ is the principal cube root of z) is

(i) $\sqrt{-1+i}$

(ii) $\sqrt{2}$

(iii) $\sqrt{3}$

(iv) $\sqrt{1+i}$.

(b) The nature of the roots of the equation $x^5 - 5x + 2 = 0$ is

(i) two complex roots, three real roots.

(ii) three real roots, one positive and two negative.

(iii) five real roots, one negative and four positive.

(iv) two complex roots, three real roots, one negative and two positive.

(c) The equation $\frac{x^3+7}{x^2+1}=5$ has

(i) no solution in [0, 2]

(ii) exactly two solutions in [0, 2]

(iii) exactly one solution in [0, 2]

(iv) all the solutions in [0, 2].

(d) The set of real values of x satisfying the inequality $x^2 + x - 6 < 6$ is

(i) $(-4, 3)$

(ii) $(-\infty, -4)$

(iii) $(3, \infty)$

(iv) none of these.

(e) The inverse of the function $f : R \to \{x \in R : x < 1\}$ given by $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is

(i) $\dfrac{1}{2}\log\dfrac{1+x}{1-x}$

(ii) $\dfrac{1}{2}\log\dfrac{2+x}{2-x}$

(iii) $\dfrac{1}{2}\log\dfrac{1-x}{1+x}$

(iv) none of these.

(f) An equivalence relation ρ on $\mathbb{Z}$ is defined by "$x\rho y \Leftrightarrow x^2 - y^2$ is divisible by 5". The equivalence classes are

(i) $\bar{0}, \bar{1}, \bar{2}$

(ii) $\bar{0}, \bar{3}, \bar{4}$

(iii) both (i) and (ii)

(iv) none of these.

(g) The total number of divisors of 360 is

(i) 23, (ii) 25, (iii) 24, (iv) none of these.

(h) If $i^n \omega^{2n} = 1$ then n is multiple of

(i) 6, (ii) 10, (iii) 12, (iv) 9.

where i and ω are usual symbol of complex number

(i) If A and B are two matrices such that AB = A and BA = B, then B^2 is equal to

(i) B, (ii) A, (iii) I, (iv) O.

(j) The solution of the system of linear equations

$$x_1 + x_3 = 1$$
$$4x_1 - x_2 + 5x_3 = 1$$
$$2x_1 + 6x_3 = 0$$

is,

(i) (1, 0, 1)

(ii) unique

(iii) more than one

(iv) exactly two.

2. Answer **one** question from (a) and (b), **one** question from (c) and (d); and **one** question from (e) and (f).

(a) Find the complete solution of the following difference equation : $x_{n+2} + 4x_n = 0$. 5

(b) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then form the equation whose roots are $\alpha+\frac{1}{\alpha}, \beta+\frac{1}{\beta}, \gamma+\frac{1}{\gamma}$. 5

(c) (i) Let $S=\{x\in R : -1<x<1\}$ and $f:R\to S$ be defined by $f(x)=\frac{x}{1+|x|}$, $x\in R$, show that f is bijective. 3

(ii) Let $f:A\to B$ be an onto mapping and S, T be two subsets of B. Then prove that,

$f^{-1}(S\cup T) = f^{-1}(S)\cup f^{-1}(T)$. 2

(d) Solve the following system of linear congruences :

$x \equiv 4 \pmod{12}$, $x \equiv 7 \pmod{21}$, $x \equiv 10 \pmod{15}$. 5

(e) For what value of a and b the following system of equations has (i) unique solution (ii) no solution (iii) more than one solution over the field of rational numbers. The system of equations are : 5

$$\begin{aligned} x_1+4x_2+2x_3 &= 1 \\ 2x_1+7x_2+5x_3 &= 2b \\ 4x_1+ax_2+10x_3 &= 2b+1. \end{aligned}$$

(f) Obtain a row echelon matrix which is row equivalent to 5

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}.$$

3. Answer ***any three*** questions, taking at least ***one*** from (a) and (b); and at least ***one*** from (c) and (d).

(a) (i) Show that the equation $\tan\left(i\log\frac{x-iy}{x+iy}\right)=2$ represents a rectangular hyperbola $x^2-y^2=xy$. 5

(ii) Let a, b, c be positive real numbers. Show that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b+c}{\sqrt[3]{abc}}$. 5

(b) (i) Solve the linear difference equation $u_{x+2}-3u_{x+1}+2u_x=e^x+e^{4x}$. 5

(ii) Solve the equation $x^4 + 12x - 5 = 0$, by Ferrari's method. 5

(c) (i) Let $f: A \to B$ be an injective mapping with $|A| = 5$. A relation ρ is defined on A by "$x \rho y$ if and only if $f(x) = f(y)$, $x, y \in A$".

Show that ρ is an equivalence relation. How many equivalence classes are there? Justify. 5

(ii) A is a non-empty set and ρ is a relation on A. Prove that ρ is an equivalence relation if and only if ρ is reflexive and $a \rho b$, $b \rho c \Rightarrow c \rho a$, for $a, b, c \in A$. 3

(iii) Let $f: A \to B$ be a bijective mapping. Then prove that $f^{-1}: B \to A$ is also bijective. 2

(d) (i) State and prove Chinese remainder theorem. 5

(ii) Prove that the number of primes is infinite. 2

(iii) If p be a prime and k be a positive integer, then prove that $\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$. 3

(e) (i) Define rank of a matrix. Find all real values of λ for which the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$$ is 2. 1+4

(ii) Let S be the set of all positive divisors of 60. On S, define a relation '$\leq$' by $a \leq b$ if and only if $a \mid b$. Prove that $(S, \leq)$ is a poset. Is $(S, \leq)$ a linear ordered set? Justify your answer. 4+1