2021

MATHEMATICS — HONOURS

Seventh Paper

(Module - XIV)

[Group - A and B]

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Probability]

(Marks : 30)

Answer ***any two*** questions.

1. (a) If we place 20 balls to 20 slots where both the balls and slots are uniquely numbered, what is the probability that no ball is placed in slots matching with their respective numbers?

(b) Among the relations defined on a set with n elements, a relation is chosen at random. What is the probability that the relation is reflexive?

(c) From an urn containing 4 white, 5 black and 2 red balls, three balls are chosen. What is the probability that they will be of (i) three different colours; (ii) same colours?

(d) The chance that a doctor will diagnose a certain disease correctly is 60%. The chance that a patient will die under his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of the doctor who had the disease died. What is the chance that his disease was diagnosed correctly? 5+3+(2+2)+3

2. (a) A straight line is divided at random into three parts. Find the chance that they can be put together to form a triangle.

(b) When is a random variable said to be continuous? Determine the value of k such that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} kx(1-x), & \text{if } 0 < x < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

is a probability density function of a random variable. Find the corresponding distribution function. Also compute $P\left(X > \frac{1}{2}\right)$.

(c) Calculate the expected values of random variables having Poisson distribution and normal distribution respectively. 5+(1+2+2+1)+4

3. (a) Find the moment generating functions of Poisson (μ) and normal (m, σ) distributions respectively.

(b) For a binomial (n, p) variate, prove that

$$\mu_{k+1} = p(1-p)\left(nk\mu_{k-1} + \frac{d\mu_k}{dp}\right),$$

where μ_k is the k-th central moment. Hence find the coefficient of skewness.

(c) Prove that, if two random variables are independent, then their correlation coefficient vanishes. Is the converse true? Justify your answer. (3+3)+(3+2)+(2+2)

4. (a) Let X_1 and X_2 be two independent random variables having Poisson distribution with parameters μ_1 and μ_2, respectively. Prove that $X_1 + X_2$ is a random variable having Poisson distribution with parameter $\mu_1 + \mu_2$.

(b) Two numbers are independently chosen at random between 0 and 1. Show that the probability that their product is less than a constant k $(0 < k < 1)$ is $k\,(1 - \log_e k)$.

(c) Using Tchebychef's inequality find the probability of getting six between 40 – 80 times in a throw of an unbiased die for 1000 times.

(d) What do you understand by an asymptotically normal distribution? State the central limit theorem for equal components. 4+4+4+(1+2)

Group - B

[Statistics]

(Marks - 20)

Answer ***any one*** question.

5. (a) Explain the terms : (i) population, (ii) random sample, (iii) distribution of a sample, (iv) sample characteristics.

(b) Prove that for a random sample $(x_1, x_2, x_3, \ldots.., x_n)$ drawn from a normal (m, σ^2) population, the sample mean $\bar{X}$ follows normal $\left(m, \frac{\sigma^2}{n}\right)$ distribution.

(c) Let $(x_1, x_2, \ldots\ldots., x_n)$ be a sample of size n (> 1) from a normal (m, σ) population. Find the sampling distribution of the statistic $t = \frac{(\bar{x} - m)\sqrt{n}}{s}$, where $\bar{x}$ is the sample mean and

$$(n-1)s^2 = \sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2.$$

(d) If $y = 2x$ and $x = \frac{y}{8}$ are the two regression lines of a bivariate sample, find the correlation coefficient of the sample. 4+6+7+3

6. (a) A population is defined by the density function

$$f(x; \theta) = \frac{x^{p-1} e^{-\frac{x}{\theta}}}{\theta^p \Gamma(p)}, \ 0 < x < \infty, \ \theta > 0,$$

where p is known and $p > 0$. Drawing a sample $(x_1, x_2, ..., x_n)$ from the population, find the maximum likelihood estimate of θ. Show that the estimate is consistent and unbiased.

(b) A sample of size 150 is drawn from a population with standard deviation 15. If the sample mean is 40, test the hypothesis that the population mean is 38. Find the range for 95% confidence limits of this population mean. [Test statistic $|Z| < 1.96$]

(c) State Neyman–Pearson theorem. Apply it to construct a test of the null hypothesis $H_0 : m = m_0$ against the alternative $H_1 : m = m_1$ for a normal (m, σ) population, where σ is known and m_0, m_1 are two given real numbers with $m_1 > m_0$.

(d) 11 sample values taken at random from the measurement of fuel efficiency of cars (km/litre) are 14.2, 12.2, 13.1, 11.2, 12.4, 13.3, 14.4, 12.6, 11.4, 13.5, 14.6. Is it reasonable to believe that the standard deviation of the population of measured values is greater than 1.1? Assume that the population is normal and use 5% level of significance. Given that $P(\chi^2 > 18.307) = 0.05$, where χ^2 is a random variable having chi-square distribution with 10 degrees of freedom.

(3+2)+5+(2+4)+4

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