2021

MATHEMATICS — HONOURS

Paper : CC-4

(Group Theory - I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer ***all*** the following multiple choice questions. Each question carries **2** marks, **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Which of the following groupoids is not semigroup?

(i) (N,o), $a\ o b = ab\ \forall\ a,b \in N$ (ii) (Z,o), $a\ o b = a+b+2\ \forall\ a,b \in Z$

(iii) (Z,o), $a o b = a-b,\ a,b \in Z$ (iv) (Z,o), $a\ o b = a+b+ab\ \forall\ a,b \in Z$

(b) Let H and K be two subgroups of a group $(G, \bullet)$ such that $o(H) = 13$ and $o(K) = 7$, then $o(HK)$ is

(i) 1 (ii) 91

(iii) 13 (iv) 7

(c) Let $(G, \bullet)$ be a cyclic group of order 24. The total number of group homomorphism of G onto itself is

(i) 7 (ii) 8

(iii) 17 (iv) 24

(d) In the permutation group S_n $(n \geq 5)$, if H is the smallest subgroup containing all the 3-cycles then which of the following is true?

(i) $H = S_n$ (ii) $H = A_n$

(iii) H is abelian (iv) $o(H) = 2$

(e) Let ϕ: $(R, +) \to (R - \{0\}, \text{o})$ be a homomorphism and $\phi(2) = 3$. Then $\phi(-6)$ is

(i) $\frac{1}{3}$ (ii) $\frac{1}{27}$

(iii) -18 (iv) $\frac{1}{9}$

(f) Choose the wrong statement among the following :

(i) If in a group $(G, \bullet)$ $(ab)^2 = b^2a^2$ for all $a, b \in G$, then G is abelian.

(ii) If $(G, \bullet)$ is a finite group, then there exists $N \in \mathbb{N}$ such that $a^N = e$, for all $a \in G$.

(iii) A group of five elements is always abelian.

(iv) If $(G, \bullet)$ is a group of even order, then there exists an element $a \neq e$ such that $a^2 = e$.

(g) If $o(a) = n$ and k divides n, which of the following is always true?

(i) $o(a^{n/k}) = k$ (ii) $o(a^{n/k}) = n$

(iii) $o(a^{n/k}) = n/k$ (iv) $o(a^{n/k}) = k.n$

(h) The value of (1 2 3 4) o (2 3 5 4 6) o (3 4 5 6) is

(i) (6 1 2 4 3 5) (ii) (6 5 3)(1 2 4)

(iii) (1 2)(3 4 5 6) (iv) (3 4 5 6 1)

(i) Show that $f : (\mathbb{C}, +) \to (\mathbb{R}, +)$ defined by $f(a + ib) = a$, for all $a + ib \in \mathbb{C}$, is onto homomorphism. Then ker(f) is

(i) $\{0\}$ (ii) $\mathbb{R}$

(iii) $i\mathbb{R} = \{ib : b \in \mathbb{R}\}$ (iv) $\mathbb{C}$

(j) Let G = $(\mathbb{Z}, +)$, H = $(24\,\mathbb{Z}, +)$. Then the order of $8 + 24\,\mathbb{Z}$ in G/H is

(i) 8 (ii) 3

(iii) 16 (iv) 24

Unit - I

2. Answer ***any two*** questions :

(a) (i) Prove that the set of all odd integers forms a commutative group with respect to '*' defined by $a * b = a + b - 1 \;\forall\; a, b \in D$

(ii) Prove or disprove : "If H and K are two subgroups of a group G then HK is also a subgroup of G". 3+2

(b) (i) If S is a finite semigroup then show that there exists an element $a \in S$ such that $a^2 = a$.

(ii) Let G be a multiplicative group and let for $a, b \in G$, $a^4 = e$ and $ab = ba^2$ where e is the identity element of G. Prove that $a = e$. 3+2

(c) Give an example of a non-abelian group of order $2n$. If a group $(G, \bullet)$ has no non-trivial subgroups, show that G must be finite and of prime order. 2+3

(d) If H is a subgroup of $(G, \bullet)$, let $N(H) = \{a \in G : aHa^{-1} = H\}$. Prove that

(i) $N(H)$ is a subgroup of G.

(ii) $H \subset N(H)$. 3+2

Unit - II

3. Answer ***any four*** questions :

(a) (i) Show that the 8th roots of unity form a cyclic group. Find all generators of the group.

(ii) Give an example of an infinite group, every element of which is of finite order. 3+2

(b) (i) Let G be the set of all permutations of the positive integers. Let H be the subset of elements of G that can be expressed as a product of a finite number of cycles. Prove that H is a subgroup of G.

(ii) Let α and β belongs to S_n. Prove that $\beta\alpha\beta^{-1}$ and α are both even or both odd. 3+2

(c) (i) If H and K be two subgroups of a group G, then prove that for any $a, b \in G$, either $Ha \cap Kb = \phi$ or $Ha \cap Kb = (H \cap K)c$ for some $c \in G$.

(ii) Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 5 & 6 & 4 & 7 & 1 \end{pmatrix}$ on S_8 as a product of transpositions. 3+2

(d) (i) Let $(G, \bullet)$ be an infinite cyclic group generated by a. Prove that a and a^{-1} are the only generators of the group G.

(ii) Let G be a cyclic group of order 30 generated by a. Find the order of cyclic group generated by a^{18}. 3+2

(e) Define cosets of a subgroup H in a group $(G, \bullet)$. The set $H = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ is a subgroup of $\mathbb{Z}_{12}$. Find all cosets of H. 2+3

(f) Prove that every non-commutative group $(G, \bullet)$ of order 10 must have a subgroup H of order 5. Also, prove that $x^2 \in H$ for all $x \in G$. 5

(g) (i) Let $a(\neq 0)$, $b \in \mathbb{R}$. Define a mapping $f_{a,b} : \mathbb{R} \to \mathbb{R}$ by $f_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. Prove that $f_{a,b}$ is a permutation on $\mathbb{R}$.

(ii) Find the largest order of an element in the group S_{12}. 2+3

Unit - III

4. Answer ***any three*** questions :

(a) (i) Let H be a normal subgroup of a group $(G, \bullet)$ and $[G : H] = m$. Prove that $a^m \in H$ for all $a \in G$.

(ii) If H is a subgroup of $(G, \bullet)$ such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G. 3+2

(b) Let $(G, \bullet)$ be a group and the mapping $f : G \to G$ be defined by $f(g) = g^{-1}$, $g \in G$. Show that f is an isomorphim if and only if G is abelian. 5

(c) (i) Prove that the quotient of an abelian group is abelian. Can the quotient of a non-abelian group be abelian? Justify.

(ii) Consider the group $G = \{1, -1, i, -i\}$ with respect to usual multiplication of complex numbers and the group $H = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ with respect to usual multiplication defined on $\mathbb{Z}_8$. Is the group G isomorphic to the group H? Justify your answer. (2+1)+2

(d) Define normal subgroups of a group. Prove that a group of prime order is simple. 1+4

(e) Let $GL_n(\mathbb{R})$ be the general linear group over $\mathbb{R}$ and $SL_n(\mathbb{R})$ be the special linear group over $\mathbb{R}$. Prove that $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*$, where there $\mathbb{R}^*$ is the group under usual multiplication of real numbers. 5