

2021

## PHYSICS — HONOURS

Paper : DSE-B-2

(Advanced Statistical Mechanics)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*

## Group - A

1. Answer **any five** questions : 2×5
- (a) Consider a system with  $N$  particles and three energy states. The states are  $E_1 = 0$ ,  $E_2 = K_B T$  and  $E_3 = 2K_B T$ . The total energy of the system is  $3000 K_B T$ . Find the number of particles.
- (b) Classify the following particles according to Fermi or Bose statistics :  
 (i)  ${}^3\text{He}$ , (ii)  $\text{H}_2$  molecule, (iii)  ${}^6\text{Li}^+$  ion, (iv)  ${}^7\text{Li}^+$  ion
- (c) A classical gas of molecules, each of mass  $m$ , is in thermal equilibrium at absolute temperature  $T$ . If  $v_x$ ,  $v_y$  and  $v_z$  are the three cartesian components of the velocity of a molecule, then calculate  $\langle (v_x - v_y)^2 \rangle$ .
- (d) The Hamiltonian of a system in 1-D is  $H = ap_x^2 + bx^2 + cx$ . Find the value of  $\langle H \rangle$ .
- (e) Show that the density operator for a pure state is idempotent.
- (f) A random walker takes a step of unit length in the positive direction with the probability  $2/3$  and a step of unit length in the negative direction with probability  $1/3$ . Calculate the mean displacement of the walker after  $n$ -step.
- (g) What do you mean by coarse graining in non-equilibrium statistical mechanics?

## Group - B

Answer **any three** questions.

2. Using the fact that the Gibbs free energy  $G(N, p, T)$  of a thermodynamic system is an extensive property of the system, show that  $G = N \frac{\partial G}{\partial N}$ . Hence show that  $G = N\mu$ , where  $\mu$  represents the chemical potential of the system. 5
3. Calculate the value of  $-\frac{\partial f(\epsilon)}{\partial \epsilon}$  at  $\epsilon = \mu$ , where  $f(\epsilon)$  represents the FD distribution function. Show that at  $T \rightarrow 0$ ,  $-\frac{\partial f(\epsilon)}{\partial \epsilon}$  at  $\epsilon = \mu$  behaves as a delta function. 2+3

Please Turn Over

4. Consider free electrons in silver (Ag) at 300 K. Test whether it is a classical system or not.

Given : Density of Ag = 10.5 gm/cc.

Atomic weight = 107.9 gm.

Boltzmann constant  $k_B = 1.381 \times 10^{-23} \text{ J/K}$ .

5

5. Consider  $N$  distinguishable and non-interacting particles. The single particle energy spectrum is  $\epsilon_n = n\epsilon$ , with  $n = 0, 1, 2, \dots, \infty$  and degeneracy  $g_n = n + 1$  ( $\epsilon > 0$  is a constant). Compute the canonical partition function and the average energy.

5

6. A system has two spin states  $S = +1$  or  $S = -1$ . The system contains  $N$  number of molecules. If there is interaction among the particles write down the Hamiltonian using Ising model. Now using Bragg Williams approximation show that the Hamiltonian can be expressed as

$$-\frac{1}{2} \gamma \epsilon N m^2 - \mu B m N.$$

5

where  $m$  is the long range order parameter,  $\gamma$  is the number of nearest neighbour of the particle,  $\epsilon$  is the interaction energy and  $\mu$  is the magnetic moment of the particle.

### Group - C

Answer *any four* questions.

7. A system has two energy levels with energy 0 and  $\epsilon$ . The system may be either unoccupied or occupied by a single particle in any one of its energy level. Calculate the grand canonical partition function of the system. Determine the average occupancy  $\langle N \rangle$  of the system. Find an expression for thermal average energy of the system.

4+3+3

8. (a) Consider a spin  $\frac{1}{2}$  system with a pure state

$$|\alpha\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- (i) find corresponding density matrix

- (ii) find the density matrix for 'up' spin state  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

- (b) The Hamiltonian of an electron in a magnetic field  $\vec{B}$  is given by  $H = -\mu_B \vec{\sigma} \cdot \vec{B}$ , where  $\vec{\sigma}$  is Pauli spin operator and  $\mu_B$  stands for the Bohr magneton. Construct the density matrix in the diagonalized representation of  $\sigma_z$ . Also calculate the  $\langle \sigma_z \rangle$ . the  $z$ -axis being taken along the field direction.

(2+2)+3+3

9. (a) Show that the pressure of the weakly degenerate bosons is less than that of the classical gas.  
[Given the relation

$$pV = Nk_B T \frac{g_{5/2}(z)}{g_{3/2}(z)}$$

where

$$g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \dots$$

- (b) Experimentally Bose–Einstein condensation was produced in a vapour of  $^{87}\text{Rb}$  atoms at a number density of  $2.5 \times 10^{12} \text{ cm}^{-3}$ . Calculate the critical temperature.
- (c) Explain why Bose condensation is not possible in one dimension. 3+3+4
10. (a) Consider a gas of free electrons confined inside a three-dimensional box. The  $z$ -component of the magnetic moment of each electron is  $\pm \mu_B$ . In the presence of a magnetic field  $B$  pointing in the  $z$ -direction, each ‘up’ state acquires an additional energy of  $-\mu_B B$ , while each ‘down’ state acquires an additional energy of  $+\mu_B B$ .
- (i) Explain why you would expect the magnetization of a degenerate electron gas to be substantially less than that of the paramagnets for a given number of particles at a given field strength.
- (ii) Obtain an expression for the net magnetization of this system at  $T = 0$ .
- (b) Consider a degenerate electron gas in which essentially all of the electrons are highly relativistic ( $\epsilon \gg mc^2$ ), so that their energies are  $\epsilon = pc$ . Calculate the chemical potential of such a system. (2+5)+3
11. (a) Calculate the fluctuations  $\langle N^2 \rangle - \langle N \rangle^2$  in number density  $N$  in grand canonical ensemble. Hence evaluate the relative root mean square fluctuation in  $N$ .
- (b) Obtain the partition function of an ideal gas in grand canonical ensemble. Hence calculate the internal energy and grand potential of the system. (4+1)+(2+2+1)
12. (a) Estimate the Fermi energy and Fermi temperature of a white dwarf star. Given : mass of proton  $9.1 \times 10^{-24} \text{ gm}$ , mass density of star  $\sim 10^7 \text{ g/cc}$ .
- (b) Explain why the electron gas in white dwarf star is highly degenerate.
- (c) Consider a person walking randomly in 1D from a point. Find the probability that the person after  $N$  displacements will be at a distance  $x = ml$ , where  $m$  is an integer and  $l$  is the step size. 10