2022

MATHEMATICS — HONOURS

Paper : CC-3

(Real Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}, \mathbb{R}, \mathbb{Q}$ denote the set of all natural, real and rational nos.

Notations and symbols have their usual meanings.

1. Answer *all* the following multiple choice questions. For each question 1 mark for choosing correct option and 1 mark for justification. 2×10

(a) Let $A = [5, 6)$ and $B = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Let $S = \{x - y : x \in A,\ y \in B\}$. Then Inf S is

(i) 3 (ii) 4

(iii) 5 (iv) 6.

(b) Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0, 2\}$ and $T = \bigcup_{n=1}^{\infty}\left(2 - \frac{1}{n+1}, 3 + \frac{1}{n}\right)$. Then $S \cap (\mathbb{R} - T)$ is

(i) open (ii) closed

(iii) both open and closed (iv) neither open nor closed.

(c) Let $A = \{[x] : 0 < x < 100\}$ and $B = \{2^i : i \in \mathbb{Z}\}$. Then $A \cup B$ is

(i) uncountable (ii) enumerable

(iii) finite (iv) empty.

([x] denotes the largest integer not exceeding x)

(d) $\left\{\frac{5^n}{n!} + \left(\frac{3}{5}\right)^n\right\}$

(i) converges to 1 (ii) converges to 0

(iii) diverges to $+\infty$ (iv) converges to 3.

(e) Number of subsequential limits of $\left\{\dfrac{(-1)^{2n}\sin\frac{n\pi}{3}}{n^5}\right\}$ is

(i) 1 (ii) 2

(iii) 3 (iv) 5.

(f) Let S be a bounded set of real numbers such that S does not have a least element. Then

(i) Inf $S = -\infty$ (ii) each point of S is an isolated point

(iii) S has at least one limit point (iv) S fails to have any limit point.

(g) Let S be a non-empty subset of $\mathbb{R}$, which of the following statement is true?

(i) If x is a boundary pt. of S then $x \in S$.

(ii) If x is a limit pt. of S then $x \in S$.

(iii) If x is an isolated pt. of S then $x \in S$.

(iv) If x is an exterior point of S then $x \in S$.

(h) Let $\{x_n\}_{n=1}^{\infty} = \{\sqrt{1}, -\sqrt{1}, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, ...\}$ and $z_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \forall n \in \mathbb{N}$. Then $\{z_n\}_{n=1}^{\infty}$ is

(i) unbounded above (ii) monotonic

(iii) bounded but not convergent (iv) convergent.

(i) Let $A = \left\{\dfrac{2}{z+1} : z \in (-1, 1)\right\}$. Then $A^d \setminus A$ is

(i) ϕ (ii) $(1, \infty)$

(iii) $\{1\}$ (iv) none of these.

(j) If $\{a_n\}_{n=1}^{\infty}$ is a monotone increasing sequence of real numbers and bounded above then the sequence $\left\{\dfrac{\sum_{i=1}^{n} a_i}{n}\right\}_{n=1}^{\infty}$ is

(i) bounded but not convergent (ii) always convergent

(iii) always divergent (iv) none of these.

Unit - 1

Answer ***any four*** questions.

2. State Archimedean property of real numbers. Use it to show that between any two distinct real numbers there are infinitely many rational numbers. 1+4

3. Prove or disprove the following statements :

   (a) If S, T are non-empty bounded sets of real numbers and $V = \{xy : x \in S,\ y \in T\}$, then $\text{Sup}V = \text{Sup}S \times \text{Sup}T$.

   (b) The set $A = \{x \in \mathbb{R} : x + y \in \mathbb{Q} \text{ for some } y \in \mathbb{R}\}$ is countable. 3+2

4. (a) Show that union of two enumerable sets of real numbers is enumerable.

   (b) Prove or disprove : If S is a set of real numbers with its derived set consisting of exactly one point, then S must be bounded. 3+2

5. (a) Prove or disprove : Every bounded infinite subset of $\mathbb{R}$ has an interior point.

   (b) Let a and b be two irrationals such that $a < b$. Show that there is a rational number q such that $a < q < b$. 2+3

6. (a) Define closed set. Give an example of a closed set which is non-empty and has no limit point in $\mathbb{R}$. (1+1)+3

   (b) Prove or disprove : $\mathbb{R} \setminus \{x \in \mathbb{R} : \sin x = 0\}$ is an open set.

7. Prove that the derived set of any set in $\mathbb{R}$ is a closed set. Hence, show that $\{x \in \mathbb{R} : x^2 - 3x + 2 \leq 0\}$ is a closed set. 3+2

8. (a) Prove or disprove : The set A of all open intervals with irrational end points is an uncountable set.

   (b) Prove or disprove : Let A and B be any two subsets of $\mathbb{R}$. If $\inf(A) \subseteq \inf(B)$, $A^d \subseteq B^d$ and $\bar{A} \subseteq \bar{B}$ then $A \subseteq B$. 3+2

Unit - 2

Answer ***any four*** questions.

9. 'l' is a limit point of a set $S \subseteq \mathbb{R}$ if and only if there exists a sequence of distinct elements of S converging to 'l'. Establish this result. 5

10. Show that every monotonically increasing sequence which is bounded above is convergent.

    Use this result to show that $\{x_n\}$ is convergent where $x_1 = \sqrt{13}$ and $x_n = \sqrt{13 + x_{n-1}}\ \forall n \geq 2$. 3+2

11. (a) Let $\{x_n\}$, $\{y_n\}$ be convergent sequence of real numbers such that $x_n \leq y_n\ \forall n \in \mathbb{N}$. Prove that $\lim\limits_{x \to \infty} x_n \leq \lim\limits_{x \to \infty} y_n$.

(b) Prove that $\left\{\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2}\right\}$ converges to zero. 2+3

12. (a) Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=l<1$. Show that $\lim_{n\to\infty}a_n=0$.

(b) Prove or disprove : If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers such that $\{x_n y_n\}$ is convergent, then both $\{x_n\}$ and $\{y_n\}$ are bounded. 3+2

13. (a) Prove or disprove : A sequence of irrational numbers can not have a rational limit.

(b) Find the limit, if exists, of the sequence $\left\{\frac{x^n}{n!}\right\}_{n=1}^{\infty}$ where $x \in \mathbb{R}$. 3+2

14. State and prove Cauchy's general principle of convergence. 5

15. (a) Prove or disprove : Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and $\lambda = Sup\{a_n : n \in \mathbb{N}\}$.

Then there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\underset{k\to\infty}{Lt}\, a_{n_k}=\lambda$.

(b) If $|a_{n+1}-a_n|<\left(\frac{1}{2}\right)^n$ for all $n \in \mathbb{N}$, show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. 3+2

Unit - 3

Answer ***any one*** question.

16. State and prove Leibnitz test. Using it show that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent. 1+3+1

17. (a) State Cauchy's n-th root test. Use it to show that the series $\frac{1^3}{3}+\frac{2^3}{3^2}+1+\frac{4^3}{3^4}+\ldots+\frac{n^3}{3^n}+\ldots$ is convergent.

(b) Show that the series $\frac{1}{5}+\frac{1}{7}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{5^3}+\frac{1}{7^3}+\ldots+\frac{1}{5^n}+\frac{1}{7^n}+\ldots$ is convergent. (1+2)+2