2022

MATHEMATICS — HONOURS

Paper : DSE-B(2)-1

(Point Set Topology)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all multiple choice questions. For each question, **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Consider the set $\mathbb{R}$ of real numbers. Let τ be the lower limit topology and σ be the upper limit topology on $\mathbb{R}$. Then

(i) $\tau \subseteq \sigma$ (ii) $\sigma \subseteq \tau$

(iii) $\tau = \sigma$ (iv) τ and σ are non-comparable.

(b) Let (X, τ) be a topological space and $A \subseteq X$. It is false that if

(i) (X, τ) is compact, then (A, τ_A) is compact

(ii) (X, τ) is T_2, then (A, τ_A) is T_2

(iii) (X, τ) is first countable, then (A, τ_A) is first countable

(iv) (X, τ) is T_1, then (A, τ_A) is T_1.

(c) Let (X, τ) be a topological space such that for every point $p \in X$, the sequence $\{p, p, p, ...\}$ has a unique limit p, then (X, τ) is

(i) T_1 space (ii) T_2 space

(iii) first countable space (iv) compact space

(d) A connected subset G of the real line $\mathbb{R}$ with at least two points must be

(i) a finite set (ii) a bounded set

(iii) an infinite closed set (iv) an uncountable set.

(e) Let $f : (X, d) \to (Y, d_1)$ be a continuous bijection where (X, d) is a compact metric space, (Y, d_1) is any metric space. Then which of the following is true?

(i) f is a homeomorphism

(ii) f^{-1} is open but not continuous

(iii) f^{-1} is closed but not continuous

(iv) f^{-1} is continuous but neither open nor closed.

(f) Let $Y = [0, 1] \cup (2, 3)$ be endowed with the subspace topology of $\mathbb{R}$. Then which of the following is true?

(i) $(2, 3)$ is open but not closed in Y

(ii) $[0, 1]$ is closed but not open in Y

(iii) $(2, 3)$ and $[0, 1]$ both are clopen in Y

(iv) $(2, 3)$ is clopen but $[0, 1]$ is not clopen in Y.

(g) In the subspace topology on $[-1, 1]$, which of the following set is open?

(i) $\{x \in \mathbb{R} : \frac{1}{2} \leq |x| \leq 1\}$ (ii) $\{x \in \mathbb{R} : \frac{1}{2} < |x| \leq 1\}$

(iii) $\{x \in \mathbb{R} : \frac{1}{2} \leq |x| < 1\}$ (iv) $\{x \in \mathbb{R} : -1 \leq x \leq \frac{1}{2}\}$

(h) Let $(\mathbb{R}, \tau_f)$ be the cofinite topological space. Then the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is

(i) a closed set (ii) an open set

(iii) both open and closed (iv) a dense set.

(i) $\mathbb{R}$ is endowed with the topology defined by $\tau = \{A \subseteq \mathbb{R} : 1 \in A\} \cup \{\phi\}$, then the derived set of $\{1\}$ is

(i) ϕ (ii) $\{1\}$

(iii) $\mathbb{R} \setminus \{1\}$ (iv) $\mathbb{R}$.

(j) In a topological space (X, τ), A is a dense subset of X and B is dense in A, then B is a

(i) open subset of X (ii) closed subset of X

(iii) dense subset of X (iv) none of the above.

Unit - 1

(Marks : 20)

Answer ***any four*** questions.

2. (a) Consider the set $\mathbb{N}$ of natural numbers and let $A_n = \{n, n+1, n+2, ...\}$. Show that the collection $\{A_n : n \in \mathbb{N}\} \cup \{\phi\}$ is a topology on $\mathbb{N}$.

(b) Find the derived set of the set $\{1947\}$ in the above topological space. 3+2

3. (a) Prove that every infinite subset of X is dense in X with respect to the cofinite topology.

(b) If D is dense in a space X and U is an open set in X, then show that $\bar{U} = \overline{U \cap D}$. 2+3

4. (a) Let $\{\tau_\alpha : \alpha \in \Lambda\}$ be a collection of topologies on a set X. Show that there is a unique smallest topology on X containing all the topologies τ_α.

(b) Let (X, d) be a metric space and $A \subseteq X$. Prove that $\bar{A} = \{x \in X : d(x, A) = 0\}$. 3+2

5. (a) Show that the collection $\mathcal{C} = \{[a, b) : a < b,\ a, b \in \mathbb{Q}\}$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

(b) Consider the order topology on the set of natural numbers, $\mathbb{N}$. Is the topology same as the discrete topology on $\mathbb{N}$? Justify. 3+2

6. (a) Consider the following collections of subsets of the set $\mathbf{R}$:

$\beta_1 = \{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) \setminus A : a, b \in \mathbb{R}$ and $A = \{\frac{1}{n} : n \in \mathbb{N}\}\}$

$\beta_2 = \{(a, \infty) : a \in \mathbb{R}\}$

Show that β_1 and β_2 are basis for some topologies on $\mathbb{R}$.

(b) Correct or Justify : $\mathbb{R}$ with usual topology and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with usual subspace topology are homeomorphic. 3+2

7. (a) Prove that if Y is a closed subset of a topological space (X, τ), then a subset G of Y is closed in the subspace (Y, τ_Y) if and only if it is closed in (X, τ).

(b) Prove that boundary of a finite set A in $(\mathbb{R}, \tau_f)$ is A itself, where τ_f denotes the cofinite topology on $\mathbb{R}$. 3+2

8. (a) What is metric invariant? Correct or Justify : A metric invariant is also a topological invariant.

(b) X is a metric space with metric d. Show that $d : X \times X \to \mathbf{R}$ is continuous. 3+2

Unit - 2

(Marks : 10)

Answer ***any two*** questions.

9. (a) Show that every metric space is a first countable space but not necessarily second countable.

(b) Let X be an uncountable set and p be a fixed point in X. Consider the topology $\tau = \{A \subseteq X : p \in A\} \cup \{\phi\}$ on X. Examine whether (X, τ) is a second countable space. 2+1+2

10. Let (X, τ) be a first countable space and $f : X \to Y$ be any function (Y being any topological space) such that for any sequence $\{x_n\}$ converging to any point $x \in X$, the sequence $\{f(x_n)\}$ converges to $f(x)$. Prove that f is continuous on X. 5

11. Let $f : X \to Y$, $g : X \to Y$ be two continuous functions from a topological space (X, τ) to a Hausdorff space (Y, σ). Prove that

(a) $F = \{x \in X : f(x) = g(x)\}$ is a closed set

(b) $f|_D = g|_D \Rightarrow f = g$, where $\bar{D} = X$. 3+2

12. (a) A G_δ set in a space X is a set that equals a countable intersection of open sets of X. Show that if X is a first countable T_1-space, every singleton set is a G_δ set.

(b) Prove that $\mathbb{R}$ endowed with cofinite topology is not a first countable space. 3+2

Unit - 3

(Marks : 15)

Answer ***any three*** questions.

13. (a) $(\mathbb{R}, \tau_C)$ is the co-countable topological space. Is the set $[0, 1]$ a compact subspace of $\mathbb{R}$? Justify.

(b) Prove or Disprove : Every infinite compact subset of $\mathbb{R}$ is connected. Is the converse true? Justify. 2+3

14. Prove that the set of components of a topological space forms a partition of that space. 5

15. Let (X, τ) be any topological space and $\beta = \{X \setminus K : K$ is compact and closed in $(X, \tau)\}$. Prove that β is a basis for some topology τ' on X such that $\tau' \subseteq \tau$. Prove that (X, τ') is compact. 3+2

16. (a) (X, τ) is a topological space and $A \subseteq X$, C is a connected subset of X that intersects both A and $X \setminus A$. Prove that C intersects boundary of A.

(b) $f : [0, 1] \to [0, 1]$ is a continuous function. Show that there exists $C \in [0, 1]$ such that $f(C) = C$, where $[0, 1]$ is endowed with the usual subspace topology. 2+3

17. Prove that the union of any family of connected sets every pair of which has an element in common, is a connected set in any topological space. Is the intersection of two connected sets always connected? Justify. 3+2