

2022

X(6th Sm.)-Mathematics-H/CC-13/CBCS

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

*The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words
as far as practicable.*

[\mathbb{N} , \mathbb{R} , \mathbb{C} denote the set of all natural, real and complex numbers respectively.]

(Notations and symbols have their usual meanings.)

1. Answer **all** the following multiple choice questions. For each question 1 mark for choosing correct option and 1 mark for justification. 2×10

(a) Which one of the following is not a metric on \mathbb{R} ?

(i) $d(x, y) = |x - y|$

(ii) $d(x, y) = |x^2 - y^2|$

(iii) $d(x, y) = |x^3 - y^3|$

(iv) $d(x, y) = \frac{|x^3 - y^3|}{1 + |x^3 - y^3|}$

(b) Let (X, d) be a metric space and $A, B \subseteq X$. Choose the correct statement.

(i) $(A \cap B)^\circ = A^\circ \cap B^\circ$

(ii) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(iii) $A \cap \partial A = \phi$

(iv) $d(A \cup B) = d(A) + d(B)$.

[$\partial(A)$ denotes boundary of A , $d(A)$ denotes diameter of A .]

(c) Let (X, d_1) and (Y, d_2) be two metric spaces and $f : X \rightarrow Y$ be a continuous mapping. Then

(i) $f(\overline{A}) = \overline{f(A)}$ for all $A \subseteq X$

(ii) $f(F)$ is closed when $F \subseteq X$ is closed in X

(iii) $f^{-1}(F)$ is closed in X whenever F is closed in Y

(iv) $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$ for any $B \subseteq Y$.

Please Turn Over

- (d) Let $A = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\} \cup \{0\}$. Then A as subspace of the real line \mathbb{R} is
- (i) compact but not complete (ii) complete but not compact
 (iii) compact and complete (iv) neither compact nor complete.
- (e) Let (\mathbb{R}, d_u) denotes the usual metric space of \mathbb{R} and (\mathbb{R}, d^*) denotes the discrete metric space of \mathbb{R} . Then which of the following statement is true.
- (i) d_u and d^* are equivalent metrics (ii) every d_u -open set is d^* -open
 (iii) $d_u = d^*$ (iv) $\{0\}$ is open in both metric spaces.
- (f) Let $f(z) = \frac{|z|}{\operatorname{Re}(z)}$, if $\operatorname{Re}(z) \neq 0$
 $= 0$, if $\operatorname{Re}(z) = 0$, then
- (i) f is continuous everywhere (ii) f is differentiable everywhere
 (iii) f is continuous nowhere (iv) f is not continuous at $z = 0$.
- (g) The radius of convergence of the power series $\sum \left(\frac{n\sqrt{2} + i}{1 + 2in} \right) z^n$ is
- (i) 1 (ii) $\sqrt{2}$
 (iii) $\frac{1}{\sqrt{2}}$ (iv) 2.
- (h) The Mobius transformation which maps $z_1 = 0, z_2 = -i$ and $z_3 = -1$ into $w_1 = i, w_2 = 1, w_3 = 0$ respectively is given by
- (i) $w = -i \left(\frac{z+1}{z-1} \right)$ (ii) $w = i \left(\frac{z+1}{z-1} \right)$
 (iii) $w = -i \left(\frac{z-1}{z+1} \right)$ (iv) $w = i \left(\frac{z-1}{z+1} \right)$.
- (i) $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1| = 1$ is
- (i) $2\pi i$ (ii) $4\pi i$
 (iii) 0 (iv) πi .

(3)

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(i) $\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz$, where C is $|z| = 4$ is

(i) 8

(ii) 10

(iii) 12

(iv) 14.

Unit - 1

(Metric Space)

Answer *any five* questions.

2. Suppose that (X_i, d_i) is a metric space for $i = 1, 2, \dots, m$ and let $X = \prod_{i=1}^m X_i$. Show that $d : X \rightarrow \mathbb{R}$

defined by $d(x, y) = \sum_{i=1}^m \frac{1}{i} (d_i(x_i, y_i))$, for all $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m) \in X$ is a metric

on X .

5

3. Let (X, d) be a metric space and Y be a subspace of X . Let Z be a subset of Y . Prove that Z is closed in Y if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$.

5

4. (a) Let $X = \left\{ (x, y) \in \mathbb{R}^2, x > 0, y \geq \frac{1}{x} \right\}$ and let d be the Euclidean metric on \mathbb{R}^2 . Check whether X is complete or not as a metric subspace of \mathbb{R}^2 .

(b) Show that uniformly continuous image of a complete metric space is not necessarily complete.

3+2

5. Suppose d is a metric on a set X . Prove that the inequality

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$

holds for all $x, y, z, w \in X$. Hence show that the function d is uniformly continuous on $X \times X$.

3+2

6. Let (X, d) be a metric space. Then prove that X is compact if and only if every collection of closed subsets of X having finite intersection property has non-empty intersection.

5

7. What do you mean by a 'contraction mapping' and a 'weak contraction mapping' of a metric space

(X, d) into itself? Let $X = [1, \infty)$ and d be the usual metric on it. Define $T : X \rightarrow X$ by $T(x) = x + \frac{1}{x}$.

Show that T is a weak contraction mapping on (X, d) but not a contraction mapping on (X, d) .

2+3

Please Turn Over

8. (a) Suppose X is a connected metric space and $f: X \rightarrow \mathbb{R}$ is continuous. If $\alpha \in (\inf f(X), \sup f(X))$, then show that $\exists z \in X$ such that $f(z) = \alpha$.
 (b) Let (X, d) be a connected metric space with X containing more than one point. Prove that X must be an infinite set. 2+3
9. (a) Find $d(A, B)$ for the sets $A = \{(x, x^2) : x \in [0, 1]\}$ and $B = \{(x, 1-x) : x \in [0, 1]\}$ with respect to the Euclidean metric on \mathbb{R}^2 .
 (b) Examine whether $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{\pi}{2} + x - \tan^{-1} x$ has a fixed point in \mathbb{R} . 3+2

Unit - 2

(Complex Analysis)

Answer *any four* questions.

10. Define stereographic projection. Show that the image of a line T under the stereographic projection is a circle minus north pole in the Riemann sphere S . 2+3
11. (a) Show that there does not exist any analytic function f such that $\text{Im}f = x^3 - y^3$.
 (b) Show that a Harmonic function $u(x, y)$ satisfies the differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$. 3+2
12. Show that the function $f(z) = e^{-z^{-4}}$ ($z \neq 0$), $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at the point $z = 0$. 5
13. Find a Mobius transformation, which maps the upper half plane $\{z : \text{Im}z > 0\}$ onto itself, fixing only $0, \infty$. 5
14. Show that the sum function of a convergent power series in z is analytic in the interior of its circle of convergence. 5
15. (a) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R > 0$. Find the radius of convergence of the

power series $\sum_{n=0}^{\infty} n^2 a_n z^n$.

- (b) Evaluate the integral $\int_{|z-i|=1} \frac{z^2 dz}{z^2 + 1}$. 3+2

(5)

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16. (a) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C . If M is a non-negative constant such that $|f(z)| \leq M$ for all points z on C at which $f(z)$ is defined,

then prove that $\left| \int_C f(z) dz \right| \leq ML$.

- (b) Without evaluating the integral show that

$$\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2} ,$$

where C is the line segment from $z = i$ to $z = 1$.

3+2