2022

MATHEMATICS — HONOURS

Paper : CC-14

(Numerical Methods)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Each of the following questions has four possible answers of which exactly one is correct. Choose the correct alternative : 1×10

(a) The relative percentage error in approximating π by 3.142 instead of 3.14159 is

(i) 0.001 (ii) 0.00041

(iii) 0.000131 (iv) 0.013

(b) Suppose a digital computer uses a number base 10, 4-bit mantissa, 2-bit exponent and 2 bits for sign, one for mantissa and one for exponent, in normalized form for floating numbers. Then which one of the following is a machine number of that computer?

(i) 3.1431×10^{-6} (ii) 10400

(iii) 10001 (iv) 3.2197×10^{2}

(c) Consider the interpolating polynomial for $f(x) = x^3$ based on the points (1, 1), (2, 8), (3, 27) and (4, 64). Find an upper bound of the interpolation error on $1 \le x \le 4$.

(i) $\frac{5}{8}$ (ii) $\frac{13}{64}$

(iii) $\frac{3^4}{4!}$ (iv) 0

(d) The trapezoidal rule of integration, when applied to $\int_a^b f(x)dx$ will give the exact value of the integral

(i) if $f(x)$ is a quadratic function of x (ii) if $f(x)$ is a linear function of x

(iii) if $f(x)$ is a cubic function of x (iv) if $f(x)$ is any function of x.

(e) If an iterative method approximately cubes the error in every three iterations then the order of convergence of the method is

(i) $\frac{1}{3}$ (ii) $\sqrt{3}$

(iii) $\sqrt[3]{3}$ (iv) 3

(f) Which of the following statements is true?

(I) Simpson's rule is exact for $f(x) = 2x + 3$

(II) Simpson's rule is exact for $f(x) = 5x^2 + 7x + 1$

(III) Simpson's rule is exact for $f(x) = 3x^4 - 5x + 10$

(i) I, II and III (ii) I and II

(iii) I (iv) None of these.

(g) If it is provided that $f(3) = 4$ is one of the initial points, what is the best choice of second point for solving by Bisection Method?

(i) -5 such that $f(-5) = -26$ (ii) 0 such that $f(0) = 5$

(iii) -3 such that $f(-3) = -2$ (iv) 13 such that $f(13) = 2$.

(h) If you have to solve $Ax = b$ many times for different vectors b but the same matrix A, it is best to

(i) compute the inverse of A (ii) use Gaussian elimination with partial pivoting

(iii) use an iterative method (iv) use LU decomposition method.

(i) Consider the initial value problem $\frac{dy}{dx} = -y$, $y(0) = y_0$. Applying second order Runge-Kutta method with step length h the value of $y(h)$ will be

(i) $\frac{y_0}{2}\left(h^2 - 2h + 2\right)$ (ii) $y_0(h-1)^2$

(iii) $\frac{y_0}{6}\left(h^2 - 2h + 2\right)$ (iv) $\frac{y_0}{6}(h-1)^2$

(j) Let $f: [0, 2] \to \mathbb{R}$ be a continuously differentiable function with $\int_0^2 f(x)dx \sim 2f(1)$.

Then the error of the approximation is

(i) $\frac{f'(\xi)}{12}$ for some $\xi \in (0,2)$ (ii) $\frac{f'(\xi)}{2}$ for some $\xi \in (0,2)$

(iii) $\frac{f''(\xi)}{3}$ for some $\xi \in (0,2)$ (iv) $\frac{f''(\xi)}{6}$ for some $\xi \in (0,2)$.

Unit – 1 and Unit – 2

(Answer ***any one*** question)

2. Using Lagrange's Interpolating polynomial, find the missing term in the following table : 5

x	0	1	2	3	4
$f(x)$	41	43	?	50	60

3. What are the basic features of Hermite interpolation formula? Prove the uniqueness of it. 2+3

Unit – 3

(Answer *any two* questions)

4. Integrate the Lagrange's interpolation polynomial

$$\phi(x) = \frac{x - x_0}{x_0 - x_1} f(x_0) + \frac{x - x_1}{x_1 - x_0} f(x_1)$$

over the interval $[x_0, x_1]$ and establish the trapezoidal rule. Give a comparison between Simpson's 1/3 rule and Simpson's 3/8 rule. 3+2

5. What do you mean by degree of precision of a mechanical quadrature formula? Prove that a necessary and sufficient condition for an $(n + 1)$ point quadrature formula with node set $\{x_0, x_1, x_2, ...x_n\}$ to have degree of precision $(2n + 1)$ is that

$\int_a^b w(x) Q_n(x)\, dx = 0$, where $w(x) = (x - x_0)(x - x_1)...(x - x_n)$ and $Q_n(x)$ is any arbitrary polynomial of degree $\leq n$. 2+3

6. Write down the quadratic polynomial which takes the same value as $f(x)$ at $x = -1, 0, +1$ and integrate to obtain

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}\left[f(-1) + 4f(0) + f(+1)\right].$$

Assuming the error to have the form $Af^{iv}(\xi), -1 < \xi < 1$, find the value of A. 5

7. Deduce numerical differentiation formula (both 1st and 2nd order) from Newton's Forward interpolation formula mentioning at least three terms. Hence find the value of first and second derivative at the left end point stating at least three terms. 3+1+1

Unit – 4

(Answer *any two* questions)

8. Show that an iterative method for computing the value of $\sqrt[k]{a}$ $(a > 0)$ is given by $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{a x_n}{(k-1)}\right]$ and also deduce that $\epsilon_{n+1} = -\frac{k-1}{2\sqrt[k]{a}} \epsilon_n^2$, where ϵ_n is the error at the nth iteration. What is the order of this iterative method? 4+1

9. Explain whether one can use Newton-Raphson method to find a real root if $f(x) = 0$ has a multiple real root? How it can be generalized and what will be its order of convergence? 2+2+1

10. Find the root of the equation $xe^x = \cos x$ using Regula-falsi method correct to three decimal places. 5

11. (a) Explain the method of fixed point iteration for approximating a simple real root α of an equation of the form $x = \phi(x)$, where $\phi(x)$ and $\phi'(x)$ are continuous on an interval about α.

(b) Derive a sufficient condition of convergence of the above method.

(c) Find also the order of convergence of the above method if $\phi'(\alpha) \neq 0$. 2+2+1

Unit – 5

(Answer ***any two*** questions)

12. For a system of n equations in n unknown $x_i\,(i = 1(1)n)$, let $x_i^{(k)}$ be the kth iterated value of the variable x_i, $x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)'$ be the solution vector obtained at the kth iteration and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$ be the exact solution of the system of equation. If $a_{ii} \neq 0$ for $i = 1(1)n$ and

$$\left[M = \max_i \left\{\frac{1}{|a_{ii}|} \sum_{j=1,\, j\neq i}^{n} |a_{ij}|\right\}\right].$$

then, prove that the estimate of the absolute error of Gauss-Seidal iteration method can be represented by $\|\xi - x^{(k+1)}\| \leq \dfrac{M}{M-1}\|\xi - x^{(k)}\|$, where $\|.\|$ denotes a norm of a vector and a_{ij} be the coefficient of i the equation in j variable x_j. 5

13. It is given that $\lambda_1 = 6$ is the largest eigenvalue and $X_1 = \begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$ is the corresponding eigenvector of

$$A = \begin{bmatrix} -306 & -198 & 426 \\ 104 & 67 & -147 \\ -176 & -114 & 244 \end{bmatrix}.$$

Find the next dominant eigen-pair of A. 5

14. Show that the matrix $E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ do not have LU decomposition. Suppose LU decomposition of the matrix $A = (a_{ij})_{n\times n}$ exists, then determine the n^2 equations in n^2 unknowns to express $A = LU$ by Crout's reduction method. If $UX = Y$ for $X = (x_1, x_2, \ldots, x_n)'$ and $Y = (y_1, y_2, \ldots, y_n)'$, then discuss how to solve $LY = b$. 2+2+1

15. Write down the basic assumptions for finding the dominant eigenvalue of a real matrix $A_{n \times n}$ by Power method. How the convergence rate of the method depends upon the magnitudes of its eigenvalues? State when the method fails. 2+2+1

Unit - 6

(Answer *any one* question)

16. Deduce Euler's method to solve initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ using Taylor's series expansion. Does this method always converge? Justify your answer. 3+2

17. Solve the equation :

$$\frac{dy}{dx} = x^2 + y^2, \; y(0) = 1,$$

by fourth order Runge-Kutta method, from $x = 0$ to $x = 0.2$, with step length $h = 0.1$. 5