2022

MATHEMATICS — HONOURS

Paper : DSE-A(2)-1

(Differential Geometry)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[The symbols used have usual meanings]

1. Answer all the following multiple choice questions. For each question, **1** mark for choosing correct option and **1** mark for correct justification. 2×10

(a) If (A_1, A_2) is a covariant vector in cartesian coordinates x^1, x^2 where $A_1 = \frac{x^1}{x^2}$ and $A_2 = \frac{x^2}{x^1}$, then the components of the vector in polar coordinates (r, θ) are

(i) $\left(\frac{\cos 3\theta + \sin 3\theta}{\cos\theta \sin\theta}, r(\sin\theta - \cos\theta)\right)$ (ii) $(\cot\theta, \tan\theta)$

(iii) $\left(\frac{\cos^3\theta + \sin^3\theta}{\cos\theta \sin\theta}, r(\sin\theta - \cos\theta)\right)$ (iv) $(r\cos\theta, r\sin\theta)$.

(b) The line element of a three-dimensional Riemannian space V_3 is $ds^2 = -(dx^1)^2 + (dx^2)^2 - (dx^3)^2$. Then the angle between two covariant vectors $(-1, 0, 1)$ and $(1, -1, 1)$ is

(i) $0°$ (ii) $90°$

(iii) $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$ (iv) $60°$.

(c) The Christoffel symbol $[12, 2]$ in a 3-dimensional Riemannian space V_3 in which the line element is given by $ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2$ is equal to

(i) $\frac{1}{x^1}$ (ii) 1

(iii) $(x^1)^2$ (iv) x^1.

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(d) In a Riemannian space V_n

$$\left(g_{hj}\, g_{ik} - g_{hi}\, g_{jk}\right) g^{hj} =$$

(i) $n\, g_{ik}$ (ii) $(n+1)\ g_{ik}$

(iii) g_{ik} (iv) $(n-1)\ g_{ik}$.

(e) If $A_i = g_{ij}\, A^j$, then $g_{ij}\, A^j{}_{,k}$ is equal to

(i) A_{ik} (ii) $A^i{}_{,k}$

(iii) $A_{i,k}$ (iv) 0.

(f) The geodesics on a right circular cylinder are

(i) circles (ii) hyperbolas

(iii) parabolas (iv) ellipses.

(g) If a surface is isometric with the Euclidean plane E^2 then

(i) $K > 0$ (ii) $K < 0$

(iii) $K = 0$ (iv) $K \neq 0$.

[where K is the Gaussian curvature of the surface]

(h) If A_j is a covariant vector, then $\dfrac{\partial A_j}{\partial x^i}$ is

(i) a (2, 0) tensor (ii) a (1, 1) tensor

(iii) not a tensor (iv) a (0, 2) tensor.

(i) The components of a contravariant vector in the x-coordinate system are 8 and 4. Then its components in the $\bar{x}$-coordinate system if $\bar{x}^1 = 3x^1$ and $\bar{x}^2 = 5x^1 + 3x^2$ are,

(i) 24 and 52 (ii) 52 and 28

(iii) 22 and 28 (iv) 24 and 56.

(j) After the contraction of a tensor of order (3, 4) becomes a tensor of order,

(i) (2, 3) (ii) (2, 4)

(iii) (3, 3) (iv) (3, 4).

Unit - 1

Answer ***any one*** question.

2. If A_{ij} is a symmetric non-singular tensor of type (0, 2) such that $A_{ij,k} = 0$ show that 5

$$\left\{ {}_{i}{}^{h}{}_{j} \right\} = \frac{1}{2} A^{hk} \left(\frac{\partial A_{ik}}{\partial x^j} + \frac{\partial A_{jk}}{\partial x^i} - \frac{\partial A_{ij}}{\partial x^k} \right).$$

3. Prove that the covariant differentiation of the fundamental tensors g_{ij}, g^{ij} and δ_j^i vanish identically. 5

Unit - 2

Answer ***any four*** questions. 5×4

4. Show that the ratio of the curvature to the torsion of a space curve different from a straight line is a non-zero constant if and only if the curve is a helix.

5. Find the osculating sphere of the curve $x^1 = 2t + 1$, $x^2 = 3t^2 + 2$, $x^3 = 4t^3 + 3$ at the point (1, 2, 3).

6. Define asymptotic lines of the surface. Prove that the parametric curves are asymptotic lines if and only if $b_{11} = b_{22} = 0$.

7. Prove that a point P of a smooth surface is umbilical (i.e. principal curvatures k_1 and k_2 are equal) if and only if the Gaussian curvature K and the mean curvature H satisfy $H^2 = K$.

8. Calculate the Gaussian curvature for a surface with metric $ds^2 = (du)^2 + c^2(dv)^2$.

9. Calculate the second fundamental form of the surface for the right helicoid given by $r = (u\cos v, u\sin v, cv)$.

10. Prove that the intrinsic derivatives of the fundamental tensors and Kronecker delta are zero.

Unit - 3

Answer ***any four*** questions. 5×4

11. When a surface is called developable? Determine whether the surface with the metric

$$ds^2 = \left(u^2\right)^2 \left(du^1\right)^2 + \left(u^1\right)^2 \left(du^2\right)^2$$

is developable or not.

12. Prove that a necessary and sufficient condition for a curve on a surface to be a geodesic is that its geodesic curvature is zero.

13. Show that the condition that the u^1-curve and u^2-curve be geodesic are $\left\{ {2 \atop 1\ \ 1} \right\} = 0$ and $\left\{ {1 \atop 2\ \ 2} \right\} = 0$ respectively.

14. Prove that $T^2 = (k - k_1)(k - k_2)$, where k is the curvature, T is the torsion and k_1, k_2 are principal curvatures.

15. Show that the geodesic curvature of the curve $u = c$ with the metric, $\lambda^2(du)^2 + \mu^2(dv)^2$ is $\dfrac{1}{\lambda\mu}\dfrac{\partial\mu}{\partial u}$.

16. Find the differential equation of the geodesic for the metric,

$$ds^2 = \left(dx^1\right)^2 + \left\{\left(x^2\right)^2 - \left(x^1\right)^2\right\}\left(dx^2\right)^2.$$

17. Using canonical geodesic equation show that the curves $u + v =$ constant are geodesics on a surface with metric,

$$\left(1 + u^2\right)du^2 - 2uv\,du\,dv + \left(1 + v^2\right)dv^2.$$