2022

MATHEMATICS — HONOURS

Paper : SEC-B-1

(Mathematical Logic)

Full Marks : 80

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct option and justify your answer : (1+1)×10

(a) Consider the sentence 'The collection of good students in Logic class form a set'. The above sentence is

(i) not a mathematical statement.

(ii) a mathematical statement whose truth value can not be determined.

(iii) a mathematical statement whose truth value is true.

(iv) a mathematical statement whose truth value is false.

(b) The proposition $(p \to q) \wedge (q \to p)$ is a

(i) tautology (ii) contradiction

(iii) contingency (iv) absurdity.

(c) Which one of the following is an adequate system of connectives?

(i) $\{\vee, \to\}$ (ii) $\{\wedge, \to\}$

(iii) $\{\sim, \wedge\}$ (iv) $\{\sim, \leftrightarrow\}$.

(d) An interpretation M is said to be model for a set of well formed formulas (wff) Γ if and only if

(i) some wff in Γ is true for M (ii) no wff in Γ is true for M

(iii) every wff in Γ is true for M (iv) None of these.

(e) The Deduction Theorem is

(i) if $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \to \beta$ (ii) if $\Gamma \cap \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \to \beta$

(iii) if $\Gamma \vdash \alpha \to \beta$ then $\Gamma \cup \{\alpha\} \vdash \beta$ (iv) if $\Gamma \vdash \alpha \to \beta$ then $\Gamma \cap \{\alpha\} \vdash \beta$.

(f) An example of the first order theories with equality is

(i) Group theory (ii) Ring theory

(iii) Ordered field (iv) Both Ring theory and Ordered field.

(g) Let $p(x, y)$ be a predicate on a non-empty set D. Then which one of the following is not a well formed formula in Predicate Calculus?

(i) $\forall x \forall y\ p(x,y)$ (ii) $\forall x \exists y\ p(x,y)$

(iii) $\exists x \forall y\ p(x,y)$ (iv) $\forall x\ p(x,y) \forall y$.

(h) Let $p(x, y, z)$ be a predicate on a non-empty set M. Then $\exists x \forall z\ p(x,y,z)$ is

(i) a statement (ii) an one-place predicate in y

(iii) a two-place predicate in x, z (iv) a three-place predicate in x, y, z.

(i) Let $p(x)$ be a predicate on a non-empty set S. Then $\sim \exists x p(x)$ is equivalent to

(i) $\forall x \sim p(x)$ (ii) $\sim \forall x\ p(x)$

(iii) $\exists x \sim p(x)$ (iv) $\sim \forall x \sim p(x)$.

(j) Consider the following two quantified predicates :

$p : \forall x(|x| = x)$ on the set of reals

$q : \exists x\left(x^2 = x\right)$ on the set of integers.

The truth values of p and q are respectively

(i) T, F (ii) F, T

(iii) T, T (iv) F, F.

Unit - I

2. Answer *any two* questions :

(a) Prove that the number of truth functions containing n variables is 2^{2^n}. 5

(b) (i) Find the converse, inverse and contrapositive of the conditional statement 'If today is Monday, then I have a mathematics test.'

(ii) Assuming the truth value of $p \to q$ be F, construct the truth table of $(\sim p \wedge q) \leftrightarrow (p \vee q)$. 3+2

(c) Let the propositions p and q be defined by

$p : 2 < 3$ (truth value 1)

q : 15 is an even integer (truth value 0).

Formulate the compound propositions using different logical connectives and display their truth values in a truth table. 5

(d) (i) Translate the given sentence into propositional formula : Either the function f is continuous or, if f is linear, then f is differentiable.

(ii) Write the general definition of formal theory. 2+3

Unit - II

3. Answer *any six* questions :

(a) (i) If $\models A$ and $\models A \to B$ then $\models B$. Prove it.

(ii) Define object language and meta language. Is English language a meta language? Justify your answer. 2+(1+1+1)

(b) When is a statement form said to be in conjunctive normal form (CNF)? Find the CNF of the formula $\sim(A \to B) \vee (\sim A \wedge C)$. 1+4

(c) Design a switching circuit corresponding to the give statement formula :

$(A \wedge B \wedge \sim C) \vee (\sim A \wedge B \wedge C) \vee (A \wedge \sim B \wedge C) \vee (A \wedge B \wedge C)$. Also find the simplest equivalent circuit. 2+3

(d) (i) Write down the axioms and rule of inferences of Propositional Calculus.

(ii) Is the following argument logically valid? Justify.

If Socrates is a man, then Socrates is mortal. Socrates is a man. Therefore, Socrates is mortal. (2+1)+2

(e) Prove that the truth function generated by any statement formula in the connectives $\sim$ and $\leftrightarrow$ only takes T an even number of times. 5

(f) Examine the logical validity of the following statement. 5

If an orange precipitate forms then either sodium or potassium is present. If sodium is not present then iron is present. If iron is present and orange precipitate forms, then potassium is not present. Hence the sodium is present.

(g) Show that $\vdash A \to A$ is a theorem in propositional logic, where A is well formed formula of propositional logic. 5

(h) State and prove Completeness Theorem in Propositional Calculus. 5

(i) Write notes on : 2+1+2

(i) Proof (ii) Theorem (iii) Logical consequence of Propositional Calculus.

(j) Prove that $\vdash (\sim B \to (B \to C))$, where B, C are well formed formulas in Propositional Calculus. 5

Unit-III

4. Answer *any four* questions :

(a) (i) Define truth set of a predicate in Predicate Calculus.

(ii) Let $\mathbb{N}$ the set of natural numbers ( ) and $D = \{1, 2, 3, 4, 5\}$ be the domain of x and y respectively. Find the truth set of the predicate $(\forall y \in D)(x + y < 12)$.

(iii) Find a counter example, if exists for the statement :

$(\forall x \in S)\left(x^2 < 80\right)$ where $S = \{5, 6, 7, 8\}$. 1+2+2

Please Turn Over

(b) Over the universe of animals, let

$A(x)$: x is a whale

$B(x)$: x is a fish

$C(x)$: x lives in water.

Translate the following in your own words :

(i) $\exists x(\sim C(x))$

(ii) $\exists x(B(x) \wedge \sim A(x))$ 1+2+2

(iii) $\forall x((A(x) \wedge C(x) \rightarrow B(x))$.

(c) Define free and bound variable in a well formed formula. Pick out the free and bound occurences of variables in the wff $\forall x_3 \left(\forall x_1 A_1^2 (x_1, x_2) \rightarrow A_1^2 (x_3, a_1)\right)$. 1+1+3

(d) Define first order theory with equality. Using this develop ordered field. 1+4

(e) State and prove Deduction Theorem in Predicate Calculus. 5

(f) (i) When is a well formed formula in Predicate Calculus said to be in Prenex Normal form?

(ii) Transform the following formula into Prenex Normal form : 1+4

$\forall x(\exists y R(x,y) \wedge \forall y \sim S(x,y) \rightarrow (\exists y R(x,y) \wedge P))$

(g) (i) Prove that the formula $\forall x \exists y A(y,x) \rightarrow \exists y \forall x A(y,x)$ is not logically valid.

(ii) Consider the following definition :

A group is non-empty set G together with a binary operation $*$ on G such that

(1) for x, y, z in G, $(x * y) * z = x * (y * z)$

(2) there is an element e of G such that

(i) $x * e = x = e * x$ and

(ii) for each x in G there is a y in G such that $x * y = e = y * x$.

Formulate the above definition as a first order theory in Predicate Calculus. 2+3