2025

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ denote the set of all natural, real, complex and rational numbers respectively.

Notations and symbols have their ususal meanings.

1. Answer all the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification : 2×10

(a) If d_1, d_2 are two metrics on a set X, then

(i) min $\{d_1, d_2\}$ is a metric on X.

(ii) $d_1.d_2$ is a metric on X.

(iii) max $\{d_1, d_2\}$ is a metric on X.

(iv) d_1^2 and d_2^2 both are metrics on X.

(b) Let A, B be subsets of a metric space (X, d). Choose the false statement.

(i) $\partial A \cap A^\circ = \varphi$ (ii) $\text{ext}(A \cup B) = \text{ext } A \cup \text{ext } B$

(iii) $d(\bar{A}, \bar{B}) = d(A, B)$ (iv) $\bar{A} = A \cup \partial A$.

(c) Let (X, d_1) and (Y, d_2) be two metric spaces and $f: X \to Y$ be a continuous mapping. Which of the following is correct?

(i) $f(\bar{A}) = \overline{f(A)}$, for all $A \subseteq X$

(ii) $f(F)$ is closed in Y when $F \subseteq X$ is closed in X

(iii) $f^{-1}(F)$ is closed in X when F is closed in Y

(iv) $f^{-1}(\bar{B}) \subseteq \overline{f^{-1}(B)}$, for all $B \subseteq Y$.

(d) Let X be an infinite set and $d: X \times X \to \mathbb{N} \cup \{0\}$ be a metric on X. Then every singleton set in (X, d) is

(i) open but not necessarily closed. (ii) closed but not necessarily open.

(iii) both open and closed. (iv) neither open nor closed.

(e) Which of the following functions is a contraction map on [0, 1]?

(i) $f(x)=x^2$ (ii) $f(x)=\sin x$

(iii) $f(x)=\dfrac{3x}{4}$ (iv) $f(x)=2x-x^2$

(f) If L is the line segment from (1, 0) to (1, 1), then $\int_L \bar{z}\,dz$ is equal to

(i) $\dfrac{1}{2}-i$ (ii) $\dfrac{1}{2}+i$

(iii) i (iv) $\dfrac{1}{2}$.

(g) The bilinear transformation which maps the points $z_1=2$, $z_2=i$, $z_3=-2$ into the points $w_1=1$, $w_2=i$, $w_3=-1$ is

(i) $w=\dfrac{5z+2}{3z+1}$ (ii) $w=\dfrac{3z+2}{iz+6}$

(iii) $w=\dfrac{iz+5}{2z+i}$ (iv) $w=\dfrac{-z+i}{3z-i}$.

(h) By Cauchy's integral formula, the value of the integral $\int_C \dfrac{dz}{z(z+\pi i)}$, where C is $|z+3i|=1$, is

(i) 1 (ii) –1

(iii) 2 (iv) –2.

(i) The radius of convergence of the series, $\sum\left(1+\dfrac{1}{n}\right)^{n^2}.z^n$ is

(i) 1 (ii) ∞

(iii) e (iv) $\dfrac{1}{e}$.

(j) Let $f(z)=\bar{z}$, then which one of the following is true?

(i) $\dfrac{d}{dz}(f(z))$ exist only at $z=0$ (ii) $\dfrac{d}{dz}(f(z))$ exist everywhere except at $z=0$

(iii) $f(z)$ is analytic everywhere (iv) $f(z)$ is non-analytic everywhere.

Unit - I

(Metric Space)

Answer ***any five*** questions.

2. Let l_2 denote the set of all real sequences $\{x_n\}$ for which $\sum_{n=1}^{\infty} |x_n^2| < \infty$ and let $d(\{x_n\}, \{y_n\}) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^2 \right\}^{\frac{1}{2}}$. Show that (l_2, d) is a metric space. 5

3. (a) Prove that a subset A of a metric space (X, d) is open if and only if $A \cap \partial A = \varphi$ (∂A = boundary of A).

(b) Let (Y, dY) be a metric subspace of a metric space (X, d). Prove that the interior of a set $A(\subseteq Y)$ in Y contains A°, the interior of A in (X, d). 3+2

4. Let (X, d_1) and (Y, d_2) be metric spaces. Show that $f: X \to Y$ is continuous if and only if $f(\bar{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$. 5

5. (a) Show that, for any subset A of a metric space (X, d), the function $f: X \to \mathbb{R}$ given by $f(x) = d(x, A)$, is uniformly continuous.

(b) Let (X, d_1) and (Y, d_2) be metric spaces and $f: (X, d_1) \to (Y, d_2)$ be uniformly continuous. Show that, if $\{x_n\}$ is a Cauchy sequence in (X, d_1) then so is $\{f(x_n)\}$ in (Y, d_2). 2+3

6. (a) Prove that every connected subset (with at least two points) in a metric space is uncountable.

(b) If $f: \mathbb{R} \to \mathbb{R}$ is a function such that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$, then prove that f cannot be continuous. 3+2

7. (a) Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X. Then show that $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

(b) Show that in a metric space any open ball is an open set. 3+2

8. (a) Prove that any continuous function on a compact metric space (X, d_1) to a metric space (Y, d_2) is uniformly continuous.

(b) Let (X, d) be a compact metric space and let A be a subset of X. Show that A is compact if it is closed. 3+2

9. Let (X, d) be a complete metric space and let f be a contraction mapping on X. Prove that there exists one and only one point x in X such that $f(x) = x$. 5

Unit - II

(Complex Analysis)

Answer ***any four*** questions.

10. (a) Find the stereographic projection of the point $z = x + iy$ on the Riemann sphere $S : x_1^2 + x_2^2 + x_3^2 = 1$.

(b) Find $\lim\limits_{z \to \frac{i\pi}{3}} \dfrac{z^3 + 1}{z^4 + z^2 + 1}$. 3+2

11. (a) Let $f : G \to \mathbb{C}$, where G is a region. Suppose the sequence $\{f(z)\}$ converges to L for every sequence of points $\{z_n\}$ in G, $z_n \neq z_0$ $(n = 1, 2, ...)$ and $z_n \to z_0$ as $n \to \infty$, then prove that $\operatorname*{Lt}\limits_{z \to z_0} f(z) = L$.

(b) Let $f(Z) = \begin{cases} \dfrac{(\operatorname{Re} Z)(\operatorname{Im} Z)}{|Z|^2}, & Z \neq 0 \\ 0, & Z = 0. \end{cases}$

Is the function $f(Z)$ continuous at $Z = 0$? Justify. 3+2

12. (a) Let G be a non-empty open connected subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be a function.

If $f'(z) = 0$ in G, prove that f is constant in G.

(b) Prove that the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = x^2 + y^2 + i\,xy$; $z = x + iy$ is nowhere analytic. 3+2

13. Let $f : G \to \mathbb{C}$, where $f(x + iy) = u(x, y) + iv(x, y)$ be a function of a complex variable on a region G. Let $u(x, y)$, $v(x, y)$ be differentiable at (x_0, y_0) and let the Cauchy Riemann equations be satisfied at (x_0, y_0), then prove that f is differentiable at $z_0 = x_0 + iy_0$. 5

14. (a) Prove that the transformation, $w = \dfrac{i(z - i)}{z + i}$ maps the upper half of z-plane into the interior of the unit circle in the w-plane.

(b) Find the bilinear transformation which transforms $z_1 = 2$, $z_2 = 1$, $z_3 = 0$ into $w_1 = 1$, $w_2 = 0$ and $w_3 = i$. 2+3

15. (a) State Cauchy's Integral formula. Using it calculate the following integral :

$$\int_C \frac{z\,dz}{(9-z^2)(z+i)},$$

where C is the circle $|z|=2$ described in positive sense.

(b) Evaluate $\displaystyle\int_C \frac{z^2-4}{z\left(z^2+9\right)}\,dz$, where C is the circle $|z|=1$. (1+2)+2

16. (a) Let $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ be a power series and $\displaystyle\sum_{n=1}^{\infty} n\,a_n z^{n-1}$ be the power series obtained by differentiating the series $\sum a_n z^n$ term by term. Then the derived series has the same radius of convergence as the original series.

(b) Find the radius of convergence of the series $\displaystyle\sum \frac{2^{-n}}{1+in^2} z^n$. 3+2