

2025

## PHYSICS — HONOURS

Paper : DSE-B-2.2

(Advanced Statistical Mechanics)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.**Symbols have their usual significance.*

## Group - A

1. Answer *any five* questions :

2×5

- (a) A one-dimensional linear harmonic oscillator has energy between  $E$  and  $E + \Delta E$ . Indicate the relevant region of phase space available to the oscillator through a suitable diagram.
- (b) Consider a free particle inside a cubical box of length  $L$  having energy between  $E$  and  $E + \Delta E$ . Calculate the number of microstates.
- (c) Show that in two-dimensions, the density of states of a particle is independent of its energy.
- (d) The mean occupation number of a single particle state for the Fermions and Bosons can be expressed as

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} \pm 1}$$

Plot the functional behaviour of  $n(\epsilon)$  with  $\epsilon$  for both the cases.

- (e) Show that for a canonical system, mean energy  $\langle E \rangle = -\frac{\partial}{\partial \beta} (\ln Z)$ , where  $Z$  is the canonical partition function.
- (f) Consider the following density matrix :

$$\rho = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Does it represent a 'pure state' or a 'mixed state'?

- (g) Derive an expression for the partition function of a two Boson system with single particle energy levels  $2\epsilon$ ,  $3\epsilon$  and  $5\epsilon$ .

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## Group - B

2. Answer *any three* questions :

- (a) An isolated, ideal solid having total energy  $E$ , comprises of a linear arrangement of  $N$  number of identical oscillators of mass ' $m$ ' which oscillate with an angular frequency ' $\omega$ '.

(i) Determine the entropy of the system.

(ii) Establish the Dulong Petit law for this solid.

(iii) Discuss whether the approach used in determining the above is purely a classical one.

2+2+1

- (b) State Liouville's theorem in classical statistical mechanics for the phase space characterized by a probability density  $\rho(q, p, t)$ . Hence establish the condition required for a system to be in statistical equilibrium.

3+2

- (c) Show that the fluctuation of energy of a system in contact with a heat reservoir can be expressed as

$$\overline{(E^2)} - (\bar{E})^2 = \left( \frac{\partial^2 \ln Z}{\partial \beta^2} \right)_{N,V},$$

$Z$  being the canonical partition function of the system.

5

- (d) The wave function of an isolated system is given by  $\psi = \sum_n a_n(t) \phi_n$  where  $\{\phi_n\}$  is the complete

orthonormal set of stationary wave functions. State the postulate of equal a priori probabilities and the random phases in terms of  $a_n$ .

5

- (e) For an ideal non-relativistic system of Bosons using the methods of the grand canonical ensemble,

(i) Determine the expression of the partition function  $\Xi$ .

(ii) Determine the expression of the chemical potential  $\mu$ .

3+2

## Group - C

Answer *any four* questions.

3. (a) Derive an expression for Fermi energy in terms of particle density.

- (b) A particle moves randomly in one dimension with probability  $p$  to the right and  $1 - p$  to the left. The length of each step is  $b$ . Calculate the probability of finding the particle at a distance  $d = Nb$  after taking  $T$  steps and hence calculate  $\langle d^2 \rangle - \langle d \rangle^2$ .

3+(5+2)

4. (a) From Bose-Einstein distribution function, obtain the Planck's law of black body radiation. Hence derive Wien's law and Rayleigh-Jeans law.

(b) What do you mean by ultraviolet catastrophe?

- (c) Assuming that the Sun radiates maximum energy at  $\lambda = 6000 \text{ \AA}$ , calculate the approximate wavelength at which the human body radiates the maximum energy. Temperature of the Sun is  $6000^\circ\text{C}$ .

(4+2)+2+2

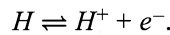
(3)

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5. (a) Calculate the grand partition function for a classical ideal gas. Given that the canonical partition function has an expression  $Z(N, V, T) = \frac{1}{N!} \frac{V^N}{\lambda^{3N}}$ , where  $\lambda = h/\sqrt{2\pi m k_B T}$  is the mean thermal wavelength of the system.
- (b) Hence calculate the mean particle number  $\langle N \rangle$  and the chemical potential of the system.

4+(3+3)

6. (a) Consider the ionization of atomic hydrogen into a hydrogen ion and an electron :



The number densities of the neutral hydrogen atoms, the hydrogen ions and the electrons at temperature  $T$  are  $n_H$ ,  $n_{H^+}$  and  $n_e$  respectively. Ignoring the excited bound states, derive the Saha ionisation equation

$$\frac{n_{H^+} n_e}{n_H} = \frac{g_{H^+} g_e}{g_H} \frac{\lambda_H^3}{\lambda_{H^+}^3 \lambda_e^3} e^{-\epsilon/k_B T},$$

where the  $g$ 's represent the statistical weights of the three species— the neutral hydrogen atoms, the hydrogen ions and the electrons. The  $\lambda$ 's represent their thermal wavelengths ( $\lambda = h/\sqrt{2\pi m k_B T}$ ) and  $\epsilon$  is the ionization energy of a hydrogen atom.

- (b) Also show that Saha's equation can be expressed

$$\frac{f^2}{1-f} = \frac{1}{n \lambda_e^3} e^{-\epsilon/k_B T},$$

where  $f = \frac{n_{H^+}}{(n_H + n_{H^+})}$  is the fraction of hydrogen atoms that are ionised. Given that

$$g_H = g_e = 2, \quad g_{H^+} = 1, \quad m_{H^+} \approx m_H \quad \text{and} \quad n_{H^+} = n_e \quad (\text{overall charge neutrality}). \quad 6+4$$

7. (a) Calculate the canonical density matrix in the momentum representation for a free particle.
- (b) The Hamiltonian of an electron in a magnetic field  $B\hat{z}$  is given by  $H = -\mu_B \vec{\sigma} \cdot \vec{B}$ , where  $\vec{\sigma}$  is Pauli spin operator and  $\mu_B$  is the Bohr magneton. Construct the density matrix in the diagonalized representation of  $\sigma_z$  and hence calculate  $\langle \sigma_z \rangle$ .

4+(3+3)

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8. (a) Consider the expression for the average magnetization in Bragg-Williams theory of the Ising model,

$$\bar{m} = \tanh \left( \frac{h + qJ\bar{m}}{kT} \right),$$

where  $h$  represents the external magnetic field strength,  $J$  is the coupling parameter between two spins in unit of energy and  $q$  is the coordination number. Consider the case of zero magnetic field ( $h = 0$ ) and show that there exists a critical temperature  $T_C$  below which the system can have a non-zero magnetization.

- (b) As a simplified model of a binary alloy, consider a square lattice of atoms which can be either of type-1 or type-2. Set spin values  $s_1 = +1$  and  $s_2 = -1$  and let there be  $N_1$  number of type-1 atoms and  $N_2$  number of type-2 atoms, such that  $N_1 + N_2 = N$ . Let the interaction energies between two neighbouring atoms be  $\epsilon_{11}$ ,  $\epsilon_{22}$  and  $\epsilon_{12}$  and there be  $N_{11}$ ,  $N_{22}$  and  $N_{12}$  bonds of each type. The energy of the binary alloy is thus  $E = \epsilon_{11}N_{11} + \epsilon_{22}N_{22} + \epsilon_{12}N_{12}$ .

Show that the above expression for  $E$  can be written in the following form :

$$E = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i - cN.$$

6+4