2025

MATHEMATICS — HONOURS

Paper : DSCC-5

(Theory of Real Functions)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural numbers, rational numbers and real numbers respectively.

Group - A

[Limit and Continuity of Functions]

(Marks : 45)

1. Answer ***any five*** questions :

(a) Evaluate : $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$. 3

(b) Prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. 3

(c) Apply Sandwich theorem and evaluate $\lim_{x\to 0+} \frac{\sin x}{x}$. 3

(d) A function f continuous on a bounded interval I may not be bounded on I. Justify it. 3

(e) Find the value of 'a' for which the function $f(x)=\begin{cases} x^2-1, & x<3 \\ 2ax, & x\geq 3 \end{cases}$ is continuous at the point '3'. 3

(f) $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=\begin{cases} |\sin\frac{1}{x}|, & x\neq 0 \\ 0, & x=0 \end{cases}$. Find the oscillation of f at $x=0$. 3

Or

Find the point of discontinuity of the function $f(x)=x-[x]$; $0<x<2$. Also mention the type of discontinuity of the function. 2+1

(g) Prove that the function $f(x)=\frac{1}{x^2}$; $x\in(0,1]$ is not uniformly continuous on $(0, 1]$. 3

(h) Let $f:[0,1]\to\mathbb{N}$ be a continuous function. Show that f is constant function. 3

Please Turn Over

2. Answer ***any six*** questions :

(a) Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Prove that $\lim_{x\to c} f(x) = l$ iff for every sequence $\{x_n\}$ in $D \setminus \{c\}$ converging to c the sequence $\{f(x_n)\}$ converges to l. 5

(b) State Cauchy's principle for existence of limit of a function at a point.

Let a function $f: (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Using Cauchy's principle prove that $\lim_{x\to a} f(x)$ does not exist, where $0 \le a \le 1$. 2+3

(c) (i) Let $f, g: \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = A > 0$ and $\lim_{x\to c} g(x) = \infty$ for some $c \in \mathbb{R}$. Prove that $\lim_{x\to c} [f(x)g(x)] = \infty$.

(ii) Show that $\lim_{x\to\infty} \frac{x-[x]}{x} = 0$. 3+2

(d) (i) If $f: [a, b] \to [c, d]$ is continuous at 'a' and $g: [c, d], \to \mathbb{R}$ is continuous at $f(a)$, then prove that $g \circ f: [a, b] \to \mathbb{R}$ is continuous at 'a'.

(ii) Show that $\frac{1+e^x \sin(x^3)}{101+\cos^2(x^2)e^{5x}}$ is continuous at every $x \in \mathbb{R}$. 3+2

(e) Let $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a) < k < f(b)$, then prove that there exists a point c in (a, b) such that $f(c) = k$. 5

Or

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ is an open set in $\mathbb{R}$. 5

(f) If $f: [a, b] \to \mathbb{R}$ be strictly monotonic increasing and continuous on $[a, b]$, then prove that f admits a continuous inverse function. 5

(g) Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then prove that $\{f(x_n)\}$ is also a Cauchy sequence in $\mathbb{R}$. Is it true when the function f is continuous on D? Justify your answer. 3+2

(h) (i) A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$, then show that there exists at least a point $c \in [0, 1]$ such that $f(c) = f(c + 1)$.

(ii) Prove that there exists $\theta \in \left(0, \frac{\pi}{2}\right)$ such that $\theta = \cos\theta$. 3+2

(i) Prove that the necessary and sufficient condition for a continuous function f on an open bounded interval (a, b) to be uniformly continuous on (a, b) is $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ both exist finitely. 5

Group - B

[Differentiability of Functions]

(Marks : 30)

3. Answer ***any four*** questions : 3×4

(a) Let I be an interval. If a function $f: I \to \mathbb{R}$ be such that $f'(x)$ exists and bounded on I, then prove that f is uniformly continuous on I.

(b) Show that there does not exist a function φ such that $\varphi'(x) = f(x)$ on [0, 2], where $f(x) = x - [x]$.

(c) Apply Lagrange's Mean Value Theorem to prove that $0 < \frac{1}{x}\ln\frac{e^x - 1}{x} < 1;\ \forall x > 0$.

(d) If f'' is continuous on some neighbourhood of c then prove that

$$\lim_{h\to 0}\frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

(e) Let $f: [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 1 - x^{4/5}$. Explain whether the equation $f'(x) = 0$ has any root in $(-1, 1)$.

(f) Prove or disprove : If a function $f(x)$ has an extreme value at an interior point 'c' of its domain, then $f'(c) = 0$.

Or

Show that -2 is an extreme point but 2 is not an extreme point of the function f where $f'(x) = (x + 2)^3(x - 2)^2;\ \ x \in \mathbb{R}$.

4. Answer ***any three*** questions :

(a) State and prove Rolle's Theorem. 2+4

(b) State Cauchy Mean Value Theorem. If f is differentiable on [0, 1], then show by Cauchy Mean Value Theorem that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in (0, 1). 2+4

(c) (i) $\varphi(x) = f(x) + f(1-x)$ and $f''(x) < 0\ \ x \in [0, 1]$. Prove that φ is increasing in $\left[0, \frac{1}{2}\right]$ and decreasing in $\left[\frac{1}{2}, 1\right]$.

(ii) Show that if two functions have equal derivative at every point of (a, b) then they differ only by a constant. 3+3

(d) Expand $\log(1 + x)$ in a finite series in power of x with Lagrange's form of remainder upto degree four. 6

(e) Find the maximum volume of a cylinder that can be inscribed in a sphere of radius $5\sqrt{3}$ cm. 6