

2025

MATHEMATICS — HONOURS

Paper : DSCC-8

(Group Theory - II and Ring Theory - I)

Full Marks : 75

The figures in the margin indicate full marks.

*Candidates are required to give their answers in their own words
as far as practicable.*

(All symbols used have their usual meanings.)

Group - A

[Group Theory - II]

(Marks : 40)

Answer *any eight* questions.

1. Let H be a subgroup of a group G such that $x^2 \in H$, for all $x \in G$. Show that H is a normal subgroup of G and G/H is commutative. 3+2
2. If ϕ be a homomorphism of a group G into a group G' , show that
 - (i) $\phi(a^{-1}) = \{\phi(a)\}^{-1}$, for any $a \in G$
 - (ii) $\phi(a^n) = \{\phi(a)\}^n$, for any integer n and any $a \in G$. 2+3
3. State and prove Cayley's theorem for a group. 5
4. Let G be a group and H, K be subgroups of G such that G is an internal direct product of H and K . Show that
 - (i) $G \simeq H \times K$ and
 - (ii) $G/H \simeq K$. 3+2
5. Let G be a multiplicative group of order 10, and let $f: G \rightarrow G$ be a mapping defined by $f(x) = x^9$, $\forall x \in G$. Show that if G is commutative, then f is an automorphism. Is the converse true? Justify your answer. 3+2
6. (a) If G be a non-commutative group with centre $Z(G)$, then prove that $G/Z(G)$ is non-cyclic.
(b) Prove that a non-abelian group of order 10 must have a trivial centre. 3+2

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7. Let G and G' be two groups and $\varphi : G \rightarrow G'$ be an epimorphism. Prove that $G/\text{Ker}\varphi$ is isomorphic to G' . 5
8. Let H, K be two finite cyclic groups of order m, n respectively. Prove that $H \times K$ is cyclic group if and only if $\text{g.c.d.}(m, n) = 1$. 5
9. (a) Let G be a cyclic group of order n . Prove that $|\text{Aut}(G)| = \phi(n)$, where ϕ is the Euler's phi function.
- (b) Let G be a group such that $\text{Aut}(G) = \{I_G\}$. Prove that G is a commutative group and $g^2 = e, \forall g \in G$. 3+2
10. Let G be a finite abelian group of order n and m be a positive divisor of n . Prove that G has a subgroup of order m . 5
11. Let H and K be two subgroups of a group G . If H is normal in G , prove that HK is a subgroup of G . Furthermore, if K is normal in G , then prove that HK is also normal in G . 5

Group - B**[Ring Theory - I]****(Marks : 35)**Answer *any seven* questions.

12. Show that a non-trivial finite ring having no divisor of zero is a ring with unity. 5
13. Define characteristic of a ring. Prove that the characteristic of an integral domain is either zero or prime number. 5
14. (a) Let a, b be two elements in a field F with $b \neq 0$. If $(ab)^2 = ab^2 + bab - b^2$, prove that $a = 1$.
- (b) Let R be an integral domain and $a \neq 0$ be an element of R . If $na = 0$ for some integer n , show that R is of finite characteristic. 3+2
15. Show that a finite integral domain is a field. 5
16. (a) Prove that $S = \{(a, 3b) : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{Z} \times \mathbb{Z}$.
- (b) Give an example of a finite ring R with unity and a subring S of R having no unity with proper justification. 3+2
17. Define ideal of a ring. Let R be a ring and $a \in R$. Prove that $S = \{r \in R : ra = 0\}$ is a left ideal of R . Give example to establish that it may not be a right ideal of R . 1+3+1

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18. (a) For positive integer n , prove that the ring \mathbb{Z}_n is isomorphic to the ring $\mathbb{Z}/n\mathbb{Z}$.
(b) For the ring homomorphism $f: \mathbb{Z}[x] \rightarrow \mathbb{Z}$ defined by $f(a_0 + a_1x + \dots + a_nx^n) = a_0$, find $\text{Ker}f$.
3+2
19. Let R be a commutative ring with unity. Prove that an ideal I is maximal if and only if R/I is a field.
5
20. Prove that a commutative ring with unity has no non-trivial ideals if and only if it is a field. 5
21. (a) Let ρ be a congruence on a ring R . Prove that $I = \{r \in R \mid r\rho 0\}$ is an ideal of R .
(b) Prove that $5\mathbb{Z}$ is a maximal ideal of the ring \mathbb{Z} . 3+2
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