# 2025

## MATHEMATICS — MDC

**Paper : CC-5**

**(Advanced Calculus)**

**Full Marks : 75**

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

(*Symbols have their usual meaning.*)

**Group - A**

**(Marks : 20)**

1. Answer ***any four*** questions :

(a) Prove that a convergent sequence is bounded. Is the converse true?– Justify your answer. 3+2

(b) Define a convergent sequence. Show that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges to 2. 1+4

(c) Prove that the sequence $\left(1+\frac{1}{n}\right)^n$ is a monotone increasing sequence and bounded above. 3+2

(d) Show by applying Cauchy's convergence criterion, that the sequence $\{x_n\}$ defined by

$$x_n = 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}$$

does not converge. 5

(e) Write the statement of Leibnitz's test of infinite series.

Is the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ convergent? Justify your answer. 2+3

(f) Test the convergence of the series $\sum \frac{n^2-1}{n^2+1}.x^n$, $x>0$. 5

(g) Test the convergence of the infinite series $\frac{5}{2\cdot 2\cdot 4}+\frac{7}{4\cdot 3\cdot 5}+\frac{9}{6\cdot 4\cdot 6}+\ldots$. 5

## Group - B

**(Marks : 25)**

2. Answer ***any five*** questions :

(a) Let $f : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{x \to c} f(x) = l$, then prove that $\lim\limits_{x \to c} |f(x)| = |l|$.

Is the converse true? Justify your answer. 2+3

(b) (i) Using Cauchy's criterion prove that $\lim\limits_{x \to 0} \cos\frac{1}{x}$ does not exist.

(ii) Given that $f(x) = |x|$, show that $f$ is not differentiable at $x = 0$. 3+2

(c) Evaluate :

(i) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

(ii) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ 2½+2½

(d) Let $f(x) = \begin{cases} x \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Check the continuity of $f$ at $x = 0$. 5

(e) Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$. Prove that there exists a point $c \in (0, 1)$ such that $f(c) = c$. 5

(f) (i) State Rolle's theorem. Is Rolle's theorem applicable to the function $f(x) = \tan x$ in $[0, \pi]$? — Justify.

(ii) Let $f(x) = 1 + x$ if $x \leqslant 2$

$= 5 - x$ if $x > 2$.

Test the nature of the function $f(x)$ at $x = 2$ as regards its differentiability. (1+2)+2

(g) (i) Show that the function $f(x) = |x - 1|$ is continuous at $x = 1$, but not differentiable at that point.

(ii) If $f(h) = f(0) + hf'(0) + \dfrac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find $\theta$ when $h = 1$ and

$f(x) = (1 - x)^{5/2}$. 3+2

(h) Prove that between any two real roots of $e^x \sin x = 1$, there exists at least one real root of $e^x \cos x + 1 = 0$. 5

(i) (i) State the geometrical interpretation of Lagrange's Mean Value theorem.

(ii) Show that the maximum value of $x + \frac{1}{x}$ is less than its minimum value. 2+3

**Group - C**

**(Marks : 30)**

3. Answer ***any six*** questions :

(a) Show that for $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

double limit exists at (0, 0). What can you say about both repeated limits at (0, 0)? 3+2

(b) If $f(x, y) = \sqrt{|xy|}$, find $f_x(0, 0)$ and $f_y(0, 0)$. 5

(c) State Schwarz's theorem.

If $f(x, y) = xy$ when $|x| \geqslant |y|$

$= -xy$ when $|x| < |y|$

show that $f_{yx}(0, 0) \neq f_{xy}(0, 0)$.

Which condition of Schwarz's theorem are not satisfied by $f$? 1+4

(d) If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, apply Euler's theorem to prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

Hence deduce that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$.

$\left[\text{assuming } \dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}\right]$. 2+3

(e) If $x^2y^2 + x^2 + y^2 - 1 = 0$ determines $y$ uniquely as a function of $x$ near the point (0, 1), calculate $\dfrac{dy}{dx}$ at that point. 5

(f) If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{2}$. 5

(g) Find $df(1, 2)$ if $f(x, y) = x^2 + xy + y^2 - 4\log_e x - 10\log_e y$. 5

(h) If $f(x, y)$ is continuous at $(a, b)$, then prove that the function $f(x, b)$ is continuous at $x = a$.

If $f(x, b)$ and $f(a, y)$ are continuous at $x = a$ and $y = b$ respectively, does it imply continuity of $f(x, y)$ at $(a, b)$? Justify your answer. 2+3

(i) (i) Use $\epsilon - \delta$ definition of limit to prove that $\lim\limits_{(x, y)\to(1, -1)} \dfrac{x^2 - y^2}{x + y} = 2$.

(ii) If $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{when } x^2 + y^2 \neq 0 \\ 0 & \text{when } x^2 + y^2 = 0 \end{cases}$,

show that $f_x(0, 0) = 1$ and $f_y(0, 0) = -1$. 2+3

(j) If $v = f(x, y)$ and $x = e^u \cos t$, $y = e^u \sin t$, then show that

$$\left(\frac{\partial v}{\partial u}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 = \left(x^2 + y^2\right)\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}.$$ 5

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