2025

MATHEMATICS — HONOURS

Paper : DSCC-10

(Ring Theory - II and Linear Algebra - I)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Ring Theory - II]

(Marks : 40)

1. Answer ***any five*** questions : 3×5

(a) If F is a field, then prove that $F[x]$ is an Euclidean domain.

(b) Prove that g.c.d. of $\overline{4}$ and $\overline{6}$ in $(Z_8, +, \cdot)$ exists but not unique.

(c) In $\mathbb{Z}[\sqrt{5}]$ prove that 2 is irreducible but not prime.

(d) Use Eisenstein's criterion to show that $f(x) = 2x^4 + 6x^3 - 9x^2 + 15$ is irreducible over $\mathbb{Z}$.

(e) Determine all associates of $8 + 3i$ in $\mathbb{Z}[i]$.

(f) Factorize $x^2 + 3$ as a product of irreducible elements of $\mathbb{Z}_7[x]$.

(g) Let F be a field. Prove that a polynomial of degree 2 or 3 in $F[x]$ is irreducible if and only if it has no root in F.

(h) Prove that every field is a regular ring.

2. Answer ***any five*** questions :

(a) Define an Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. 1+4

(b) In a principal ideal domain R, show that for any two non-zero elements $a, b \in R$, $m = \text{l.c.m.}(a, b)$ if and only if $\langle m \rangle = \langle a \rangle \cap \langle b \rangle$. 5

(c) Prove that [2] is a prime element in Z_{10} but [2] is not irreducible in Z_{10}. 2+3

(d) Define a unique factorization domain. Prove that in a unique factorization domain every irreducible element is prime. 1+4

(e) Prove that the polynomial $x^{p-1} + x^{p-2} + ... + x + 1$ is irreducible in $Z[x]$, where p is a prime number. 5

(f) If R is a commutative ring with unity such that $R[x]$ is P.I.D, then show that R is a field. 5

(g) Prove that any ring (with more than one element) which is commutative and without non-zero divisors of zero can be embedded in a field. 5

(h) (i) Show that a ring R with identity is a regular ring if and only if for any left ideal A and right ideal B of R, $BA = A \cap B$.

(ii) Prove or disprove : $\mathbb{Z}$ is a regular ring. 4+1

Group - B

[Linear Algebra - I]

(Marks : 35)

3. Answer ***any five*** questions : 3×5

(a) Find a basis for the vector space $\mathbb{R}^3$ that contains the vectors (1, 2, 0) and (1, 3, 1).

(b) Find the dimension of the subspace S of the vector space $\mathbb{R}^3$ defined by :

$S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y = z,\ 2x + 3z = y\}$

(c) Find the co-ordinate vector of α in $\mathbb{R}^3$ relative to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha = (0, 3, 1)$, $\alpha_1 = (1, 1, 0)$, $\alpha_2 = (1, 0, 1)$, $\alpha_3 = (0, 1, 1)$.

(d) Find a basis for the row space of the matrix $A = \begin{pmatrix} 0 & 3 & 7 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

(e) Show that two eigen vectors of a square matrix A over a field F corresponding to two distinct eigenvalues of A are linearly independent.

(f) If A and P be both $n \times n$ matrices and P be non-singular, then prove that A and $P^{-1}AP$ have same eigenvalues.

(g) Let $T : V \to W$ be a linear mapping where V are W are finite dimensional vector spaces of same dimension over a field F. If T is surjective (onto), then prove that T is one-to-one.

(h) In a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$, $[T] = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{2}{3} & 4 \end{bmatrix}$ associated with the basis {(1,0), (1,1)}. Find $T(x, y)$.

4. Answer ***any four*** questions :

(a) Let V be a finite dimensional vector space over a field F_1 and W be a subspace of V. Show that $\dim\left(V/W\right) = \dim V - \dim W$. 5

(b) When is a set of vectors $\{x_1, x_2, \ldots, x_n\}$ said to be linearly independent? Show that the set of vectors {(1, 2, 2), (2, 1, 2), (2, 2, 1)} is linearly independent in $\mathbb{R}^3$. 1+4

(c) Find a basis and find dimension of the vector subspace W of $\mathbb{R}^3$ where

$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0,\ 2x + y + 3z = 0\}$. 3+2

(d) Let $AX = 0$ be a homogeneous system of equations in n unknowns and $X(A)$ be the solution space of the system of equations. Then prove that rank of A + rank of $X(A) = n$. 5

(e) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of T relative to the ordered bases $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$. 5

(f) Prove that eigen values of a real symmetric matrix are all real. 5

(g) State Cayley-Hamilton Theorem and verify it for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$. Hence find out A^{-1}, if exists. 1+3+1
