2025

MATHEMATICS — HONOURS

Paper : DSCC-11

(Riemann Integration and Series of Functions)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Throughout the question $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of real numbers, rational numbers and set of natural numbers respectively.

Group - A

(Marks : 50)

1. Answer ***any five*** questions : 3×5

(a) Let $f : [0, 4] \rightarrow$ be defined by $f(x) = x^4 - 4x^3 + 10$ and $P = \{1, 2, 3, 4\}$. Find $U(P, f)$ and $L(P, f)$.

(b) Prove that, $\mathbb{Q}$ is a negligible set.

(c) Let $f(x) = [x]$; $0 \leq x \leq 3$. Evaluate $\int_0^3 f$.

(d) Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \lim_{n \to \infty} (\sin 2x)^n$. Check whether f is Riemann integrable on [0, 1].

(e) If f is continuous on $[a, b]$, $f(x) \geq 0$ on $[a, b]$ and $\int_a^b f = 0$, then prove that $f(x) = 0$ for all x on $[a, b]$.

(f) Let $f(x) = \begin{cases} c; & 0 \leq x \leq c \\ 2c; & c < x \leq 1 \end{cases}$. If $\int_0^1 f = \frac{7}{16}$, then find the value of c.

(g) Examine the convergence of $\int_1^2 \frac{\log x}{\sqrt{2-x}} dx$.

(h) Find the value of $\int_{-\infty}^{\infty} e^{-x^2} dx$.

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2. Answer ***any seven*** questions :

(a) Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. If Q is a refinement of P, then prove that $U(P, f) \geq U(Q, f)$ and $L(P, f) \leq L(Q, f)$. 5

(b) State and prove a necessary and sufficient condition for Riemann integrability of a bounded function f defined on $[a, b]$. 1+4

(c) (i) Let f, g, h be three bounded functions on $[a, b]$, such that $f(x) \leq g(x) \leq h(x)$, $\forall x \in [a, b]$. If $f(x)$ and $h(x)$ are Riemann integrable and $\int_a^b f(x)dx = \int_a^b h(x)\,dx$, show that g is also Riemann integrable on $[a, b]$.

(ii) Find the number of points of discontinuities of the function

$\phi(x) = \int_0^x \left[\sqrt{t}\right] dt$, $0 \leq x \leq 2026$, ($[r]$ denotes greatest integer $\leq r$). 3+2

(d) If $f, g : [a, b] \to \mathbb{R}$ are Riemann Integrable on $[a, b]$, show that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ (Assume that $f + g$ is also Riemann Integrable on $[a, b]$). 5

(e) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then prove that the function $F(x) = \int_a^x f(t)\,dt\,;\, a \leq x \leq b$ is differentiable on $[a, b]$. 5

(f) Let f be a differentiable function on $[a, b]$ with $|f'(x)| \leq C$, $\forall x \in [a, b]$ for some $C > 0$. Prove that for any partition P of $[a, b]$,

$$U(P, f) - L(P, f) \leq C\,\|P\|\,(b - a).$$ 5

(g) Prove that $\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \leq \frac{\pi}{6} \cdot \frac{1}{\sqrt{1 - \frac{k^2}{4}}}$, $k^2 < 1$. 5

(h) Let f be defined on $[-2, 2]$ by

$$f(x) = \begin{cases} 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Show that f is integrable on $[-2, 2]$. Evaluate $\int_{-2}^{2} f$, using fundamental theorem of integral calculus. 1+4

(i) Show that the improper integral $\int_0^1 \frac{1}{x} \sin\frac{1}{x}\,dx$ is convergent. 5

(j) (i) Prove that $\int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx$ is convergent if and only if $n < 1 + m$.

(ii) Find the value of $\int_0^{\infty} x^2 e^{-x^4}\,dx$. 3+2

(k) State Dirichlet's test for improper integral. Applying this show that

$\int_0^1 x^{-3/2} \sin\left(\frac{1}{x}\right)dx$ is convergent. 2+3

Group - B

(Marks : 25)

3. Answer ***any five*** questions :

(a) Prove that a sequence $\{f_n\}$ of real-valued functions defined in $E \subset \mathbb{R}$ converges uniformly in E if and only if for every $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $\forall\, m, n > k$ and $\forall\, x \in E$, $|f_n(x) - f_m(x)| < \epsilon$. 5

(b) Show that the sequence $\left\{\frac{nx}{1+n^2x^2}\right\}_n$ of continuous functions defined on [0, 1] has a continuous limit function. Also show that the convergence is not uniform. 2+3

(c) Let $f_n(x) = \frac{\log(1+n^2x^2)}{n^2}$, $x \in [0, 1]$. Show that $\{f_n\}$ is uniformly convergent on [0, 1]. 5

(d) Show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$ is uniformly convergent on $\mathbb{R}$. Hence find $\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$. 2+3

(e) Examine the uniform convergence of

$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots\ldots\ldots\ldots$, where $0 \le x \le 1$. 5

(f) Assuming $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots\ldots$; $(-1 < x < 1)$, obtain the power series expansion of $\log_e(1 + x)$ and find the region of convergence of the power series of $\log_e(1 + x)$. 3+2

(g) (i) Prove or disprove : Radius of convergence of a power series remains invariant under term-by-term differentiation.

(ii) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, where for all $n \in \mathbb{N}$

$$a_n = \begin{cases} \dfrac{1}{3^n}, & \text{if } n \text{ is odd} \\ \dfrac{1}{2^n}, & \text{if } n \text{ is even.} \end{cases}$$

3+2

(h) Find the Fourier series for a periodic function $f : \mathbb{R} \to \mathbb{R}$ of period 2π defined by $f(x) = |x|$ for $-\pi \leq x \leq \pi$. 5
