2025

MATHEMATICS — HONOURS

Paper : DSCC-3

(Real Analysis)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ denote the set of all natural, real and rational numbers respectively.

Group - A

(Marks : 30)

1. Answer ***any three*** questions :

(a) Show that the set $\left\{x \in \mathbb{R} : \sin\frac{1}{x} = 0\right\}$ is countable. 3

(b) Give example of a countable set whose derived set is uncountable. 3

(c) Prove or disprove : If each point of a set S is isolated then S cannot have limit points. 3

(d) Find the isolated points of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. 3

(e) What is limit point of a set? Find all limit points of the set $\left\{\frac{2}{x+1} : x \in (-1, 1)\right\}$ in $\mathbb{R}$. 1+2

2. Answer ***any three*** questions :

(a) (i) Let A and B be bounded non-empty subset of $\mathbb{R}$ and let $A - B = \{a - b : a \in A,\ b \in B\}$. Prove that $\inf(A - B) = \inf A - \sup B$.

(ii) Show that $(0, 1)$ is not enumerable. 3+4

(b) (i) Find $\sup A$ and $\inf A$ where $A = \{x \in \mathbb{R}: 3x^2 + 8x - 3 < 0\}$.

(ii) Examine whether the set $\{x \in \mathbb{R} : 1 < |x - 2| \leq 2\}$ is open or closed. 3+4

(c) Prove or disprove :

(i) Every non-empty bounded subset of $\mathbb{Q}$ has a lub in $\mathbb{Q}$.

(ii) $(A_1 \cap A_2)^d = A_1^d \cap A_2^d$ for any two subsets A_1 and A_2 of $\mathbb{R}$, where A^d denotes the derived set of A. 4+3

(d) (i) Prove that each interior point of a set is a limit point. Is the converse true?

(ii) Prove that the union of a finite number of closed sets in $\mathbb{R}$ is closed. Does the conclusion hold good for arbitrary union of closed sets? Justify. (2+1)+(2+2)

(e) (i) Prove or disprove : The union of an infinite number of closed sets in $\mathbb{R}$ is always a closed set.

(ii) Define closed set and prove that the intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is a closed set. 3+(1+3)

Group - B

(Marks : 35)

3. Answer ***any four*** questions : 2×4

(a) Prove or disprove : $\left\{\left(\frac{4}{5}\right)^n + \left(\frac{5}{4}\right)^n\right\}$ is a convergent sequence.

(b) Prove or disprove : If $\lim_{n\to\infty}\left(x_{n+1} - x_n\right) = 0$, then $\{x_n\}$ is a Cauchy sequence.

(c) Prove or disprove : If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} x_n y_n = 0$ for every sequence $\{y_n\}$.

(d) Find lim sup x_n, where $x_n = \left(1 - \frac{1}{n^2}\right)\sin\frac{n\pi}{2}, n \in \mathbb{N}$.

(e) Find $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$.

(f) Prove or disprove : Every bounded sequence is a Cauchy sequence.

4. Answer ***any three*** questions :

(a) (i) Give example of two non-convergent sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n y_n\}$ is convergent.

(ii) Let $\{x_n\}$, $\{y_n\}$ be two convergent sequences such that $x_n \le y_n$ for all $n \ge 2026$. Prove that $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$. 2+2

(b) Prove that every convergent sequence is bounded. Is the converse true? Justify. 2+2

(c) Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of positive real numbers.

Prove that lim sup$(x_n y_n) \le$ lim sup x_n . lim sup y_n. Cite an example for which the strict inequality holds. 3+1

(d) (i) Show that $\lim_{n\to\infty} x^n = 0$ if $|x|<1$.

(ii) Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$. If $0 \le l < 1$, then prove that $\lim_{n\to\infty} u_n = 0$. 2+2

(e) Show that a Cauchy sequence of real numbers is convergent. 4

5. Answer ***any three*** questions :

(a) Prove that every bounded sequence has a convergent subsequence. 5

(b) (i) If $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l(\neq 0)$ then prove that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$.

(ii) Prove that $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$. 3+2

(c) A sequence $\{x_n\}$ is defined by $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for $n \in \mathbb{N}$ and $x_1 > x_2 > 0$. Show that the sub-sequence $\{x_{2n-1}\}$ is decreasing and the subsequence $\{x_{2n}\}$ is increasing, both are convergent and converge to the same limit $\frac{x_1 + 2x_2}{3}$. 5

(d) (i) Show that a bounded sequence $\{u_n\}$ is convergent if and only if lim sup u_n = lim inf u_n.

(ii) Prove that $\lim_{n\to\infty} \frac{\{(n+1)(n+2)...(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}$ using Cauchy's second limit theorem. 3+2

(e) State and prove the sandwich theorem and by using this theorem prove that $\lim_{n\to\infty} \frac{3n^3 + 2n^2 + 1}{n^3 + 1} = 3$. 1+2+2

Group - C
(Marks : 10)

6. Answer ***any two*** questions :

(a) State Gauss' Test. Examine the convergence of the series $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 + \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2 + ...$ 1+4

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(b) Prove or disprove :

(i) If a series $\sum x_n$ is convergent, then $\sum x_n^2$ is also convergent.

(ii) If $\sum y_n^2$ is convergent, then $\sum y_n$ is also convergent. 2+3

(c) (i) State Raabe's test.

(ii) Show that the series $1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}+...$ is convergent. 2+3

(d) State Leibniz's test. Prove that the series $\frac{3}{1.2}-\frac{5}{2.3}+\frac{7}{3.4}-....$ converges conditionally. 1+4
