2026

MATHEMATICS — HONOURS

Paper : DSCC-13

(Metric Spaces and Complex Analysis-I)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Throughout the question $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of real numbers, rational numbers and complex numbers respectively.

Group - A

[Metric Spaces]

(Marks : 45)

1. Answer ***any five*** questions : 3×5

(a) Let X denote the set of all Riemann integrable functions on $[a, b]$. For $f,\ g \in X$ define $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$. Is '$d$' a metric on X? Justify.

(b) In a metric space (X, d) if $a, b \in X$ and $a \neq b$, then show that there exist open balls S_a and S_b containing a and b respectively, such that $S_a \cap S_b = \varphi$.

(c) Let (X, d) be a metric space. Prove that, a Cauchy sequence $\{x_n\}_n$ in X will be convergent if it has a convergent subsequence.

(d) Let (X, d) and (Y, ρ) be metric spaces and $f, g : X \rightarrow Y$ be continuous functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

(e) Prove that a totally bounded metric space is bounded.

(f) Prove that a discrete metric space with more than one point is disconnected.

(g) Let $X = \{x \in \mathbb{Q} : x \geq 1\}$ be a metric space with the usual metric $d(x, y) = |x - y|$, $x, y \in X$ and $T : X \rightarrow X$ be a mapping defined by $T(x) = \frac{x}{2} + \frac{1}{x}, x \in X$. Show that T is a contraction mapping.

2. Answer *any six* questions :

(a) Let (X, d) be a metric space, for $x, y \in X$, define
$$d^*(x,y)=\frac{d(x,y)}{1+d(x,y)}.$$
Prove that (X, d^*) is a metric space. Is it bounded metric space? Justify your answer. 4+1

(b) Let (X, d) be a metric space. A and B are two subsets of X. Prove that $d(A \cup B) \leq d(A) + d(B) + d(A, B)$, where $d(A)$ denotes the diameter of A and $d(A, B)$ denotes distance between the sets A and B. 5

(c) Let Y be a metric subspace of a metric space (X, d). Then prove that $A(\subseteq Y)$ is open in (Y, dy) if and only if $A = G \cap Y$ for some open set G in (X, d). Deduce that $[0, 1]$ is open in $Y = [0, 1] \cup (2, 3)$, Y being subspace of real line. 4+1

(d) If $\{x_n\}$ is a Cauchy sequence in a metric space (X, d), then show that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Is the converse true? Justify. 2+3

(e) Let $f: (X, d) \to (Y, \rho)$ be an uniformly continuous function and $\{x_n\}$ be a Cauchy sequence in X. Show that $\{f(x_n)\}$ is a Cauchy sequence in Y. Hence or otherwise show that $f(x)=\frac{1}{x}, x\in(0,1]$ is not uniformly continuous. (Here $(0, 1]$ is equipped with the subspace metric from the real line). 3+2

(f) Let (X, d) be a complete metric space and $\{F_n\}$ be a descending sequence of non-empty closed sets with $d(F_n) \to 0$ as $n \to \infty$. Prove that $\bigcap\limits_{n=1}^{\infty} F_n$ contains exactly one point. 5

(g) Let (X, d) be a metric space. Then prove that X is compact if and only if every collection of closed subsets of X having finite intersection property has non-empty intersection. 5

(h) Let (X, d) be a complete metric space and $T: X \to X$ be a contraction mapping. Prove that T has a unique fixed point. 5

Group - B

[Complex Analysis - I]

(Marks : 30)

3. Answer *any six* questions :

(a) Find the point $Q = (\alpha, \beta, \gamma)$ on the Riemann sphere $x^2 + y^2 + z^2 = 1$ that corresponds to the complex number $z = x + iy$, $(x, y \in \mathbb{R})$ under steriographic projection. 5

(b) Show that a function $f: \mathbb{C} \to \mathbb{C}$, defined by $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if both u and v are continuous at (x_0, y_0). 5

(c) Let $f: G \to C$ where $f(x+iy) = u(x, y) + iv(x, y)$ be a function of a complex variable on a region G. Let $u(x, y)$, $v(x, y)$ be differentiable at (x_0, y_0) and let the Cauchy-Riemann equations be satisfied at (x_0, y_0). Prove that f is differentiable at $z_0 = x_0 + iy_0$. 5

(d) Prove that the function

$$f(x+iy) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0 & , \text{ if } x^2+y^2 = 0 \end{cases}, \quad (x, y \in \mathbb{R})$$

satisfies Cauchy-Riemann equations at the origin but $f'(0)$ does not exist. 4+1

(e) (i) Let f be analytic in a region D of C. Prove that if $f'(z) = 0$ in D, then f is constant in D.

(ii) Let f be a real valued function of a complex variable and differentiable at a point $\alpha \in \mathbb{C}$. Prove that $f'(\alpha) = 0$. 3+2

(f) If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$ 5

(g) If $u(x, y) = e^x \sin y$, use Milne-Thomson method to find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic in a finite region. 5

(h) (i) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Prove that $\sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$ has radius of convergence R.

(ii) Find the radius of convergence of the power series $\sum a_n z^n$, where $a_n = \dfrac{n!}{(2n)!}$. 3+2

(i) Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $\omega = i, 0, -i$. Also find the image of $|z| \leq 1$ under this transformation. 5

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