

2026

MATHEMATICS — HONOURS

Paper : DSCC-5

(Theory of Real Functions)

Full Marks : 75

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.* $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural numbers, rational numbers and real numbers respectively.

Group – A

[Limit and Continuity of Functions]

(Marks : 45)

1. Answer **any five** questions :

3×5

- (a) Let $f(x) = x$, $x \in \mathbb{Q}$
 $= 2 - x$, $x \in \mathbb{R} \setminus \mathbb{Q}$

Show that $\lim_{x \rightarrow 1} f(x) = 1$.

- (b) Examine if $\lim_{x \rightarrow 0} \left(\sin \frac{1}{x} + x \sin \frac{1}{x} \right)$ exists or not.

- (c) Find the value of k for which the function $f(x) = \begin{cases} (1+2x)^{\frac{1}{3x}} & ; x \neq 0 \\ k & ; x = 0 \end{cases}$

is continuous at the point $x = 0$.

- (d) Show that the set $\{x \in \mathbb{R} : \cos x = 0\}$ is a closed set in $\mathbb{R}$.
- (e) Give an example of two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f + g$ is continuous on $\mathbb{R}$ but f, g both are discontinuous on $\mathbb{R}$.
- (f) Show that $\sin x^2$ is not uniformly continuous on $[0, \infty)$.
- (g) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that $f(x) = g(x) \forall x \in \mathbb{Q}$. Is $f = g$ on $\mathbb{R}$? Justify your answer.
- (h) Show that $f(x) = \tan x$ on $[a, b]$, where $-\pi/2 < a < b < \pi/2$ is uniformly continuous.

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2. Answer **any six** questions :

- (a) State and prove Sandwich Theorem on limits of real-valued functions. 5
- (b) Let f be a function on $\mathbb{R}$ such that $f(x) = \begin{cases} 1; & x \in \mathbb{Q} \\ 0; & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Prove that f is everywhere discontinuous on $\mathbb{R}$. 5
- (c) If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, then prove that f is bounded on $[a, b]$ and attains its bounds in $[a, b]$. 5
- (d) Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and injective on $[a, b]$. Prove that f is strictly monotone on $[a, b]$. 5
- (e) Prove that for any monotonically increasing function f on $\mathbb{R}$, if $a < b$, then $f(a-) \leq f(a+) \leq f(b-) \leq f(b+)$. 5
- (f) Define a Lipschitz function. Show that a Lipschitz function defined in an interval I is uniformly continuous in I . Is the converse true? Justify. 1+2+2
- (g) Let $f(x) = [x] + [-x]$, $\forall x \in [0, 5]$. Find all points of discontinuity of $f(x)$ in $[0, 5]$. Also mention the type of each discontinuity of the function. 4+1
- (h) (i) Let $f: [0, 1] \rightarrow [0, 1]$ be continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$.
(ii) Prove or disprove : There exists a monotonic function defined on $[0, 1]$ such that the function is discontinuous at every irrational point in $[0, 1]$. 3+2
- (i) If f be a function on $\mathbb{R}$ such that $f(x+y) = f(x) + f(y)$; $\forall x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, then prove that f is uniformly continuous on $\mathbb{R}$. 5

Group – B

[Differentiability of Functions]

(Marks : 30)

3. Answer **any four** questions :

3×4

- (a) Let $f: [a, b] \rightarrow \mathbb{R}$ be a function such that $f'(c) > 0$ for some $c \in (a, b)$. Prove that f is increasing at $x = c$.
- (b) Let $g: [0, 2] \rightarrow \mathbb{R}$ be defined by $g(x) = \begin{cases} 3 & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 < x \leq 2 \end{cases}$.

Does there exist a function F on $[0, 2]$ such that $F'(x) = g(x) \forall x \in [0, 2]$?

- (c) Verify Rolle's theorem for the function $f(x) = \sin \frac{1}{x}$ on $\left[\frac{1}{3\pi}, \frac{1}{2\pi} \right]$.

(d) Using Cauchy's mean value theorem, show that $\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta$, where $0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

(e) Using Taylor's theorem, show that $x - \frac{x^2}{2} < \log(1+x) < x$, for $x > 0$.

(f) Prove that $f(3)$ is a local minimum value of $f(x) = |x+2| + |3-x| + |x-5|$, $x \in \mathbb{R}$ but $f'(3)$ does not exist.

4. Answer **any three** questions :

(a) State and prove Darboux's theorem for a differentiable function. Using this theorem, show that $\cos x$ will assume every value between 0 and 1 in the interval $\left(0, \frac{\pi}{2}\right)$. 1+4+1

(b) State Rolle's theorem. If $a_0, a_1, a_2, \dots, a_n$ be real numbers satisfying the relation $\frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + a_n = 0$, show that $a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$ has a root in $(0, 1)$. 2+4

(c) (i) If $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and f is differentiable in (a, b) such that $f'(x) = 0$; $\forall x \in (a, b)$, prove that f is constant on $[a, b]$.

(ii) If a function f has extreme value at $x = a$ and $f'(a)$ exists, prove that $f'(a) = 0$. 3+3

(d) Find the Taylor's infinite series expansion of $\sin x$, about $x = 0$ together with the range of validity. 6

(e) How should a wire of length l cm, be divided into two parts so that if one part is bent into a circle, the other part into a square, the sum of the areas of the two plane figures becomes minimum? 6