

2026

MATHEMATICS — HONOURS

Paper : DSCC-8

(Group Theory - II and Ring Theory - I)

Full Marks : 75

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.**All symbols used have their usual meanings.*

Group - A

[Group Theory - II]

(Marks : 40)

Answer *any eight* questions.

1. (a) If a group G has exactly one subgroup of order n , prove that this subgroup is normal in G .
 (b) Prove or disprove :
 Every subgroup of prime index in a group is normal. 3+2
2. State Cayley's theorem. Using Cayley's theorem find out a subgroup of S_3 which is isomorphic to the group $\mathbb{Z}_3$. 2+3
3. Let G be a group. Show that the set $A = \{x \in G : \phi(x) = x, \forall \phi \in \text{Aut}(G)\}$ forms a normal subgroup of G , where $\text{Aut}(G)$ is the automorphism group of G . 5
4. Define the innerautomorphism of a group G determined by an element $a \in G$. Prove that the group of all innerautomorphisms of a group G is isomorphic to the quotient group $\frac{G}{Z(G)}$, where $Z(G)$ is the center of G . 1+4
5. State Cauchy's theorem for finite abelian group. Using the theorem, show that every abelian group of order 14 contains exactly 6 elements of order 7. 2+3
6. Prove that any two finite cyclic groups of the same order are isomorphic. 5
7. (a) Give an example of a group of order 8 which is not Abelian. Justify your answer.
 (b) "Every group of order 35 is cyclic." — Prove or disprove the statement. 2+3

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8. Suppose G is a finite Abelian group and p is a prime divisor of order of G . Prove that there is an element $a(\neq e)$ in G , such that $a^p = e$, where e is the identity element in G . 5
9. Find the number of elements of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$. 5
10. State and prove the Second Isomorphism Theorem for groups. 5
11. Let $G = S_3$, $G' = (\{-1, 1\}, \cdot)$ and $\phi : G \rightarrow G'$ is defined by

$$\phi(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is an even permutation in } S_3 \\ -1 & \text{if } \alpha \text{ is an odd permutation in } S_3 \end{cases}$$

Prove that ϕ is a homomorphism. Determine $\text{Ker } \phi$. Hence or otherwise prove that A_3 is a normal subgroup of S_3 . 1+1+3

Group - B

[Ring Theory - I]

(Marks : 35)

Answer *any seven* questions.

12. (a) Let F be a field of characteristic p . Prove that $(x + y)^p = x^p + y^p$, $\forall x, y \in F$.
 (b) Give an example of infinite ring of finite characteristic. 3+2
13. Let $\mathbb{Z}$ be the ring of integers. Define two relations ρ_1 and ρ_2 on $\mathbb{Z}$ by
- $$a \rho_1 b \Leftrightarrow a \equiv b \pmod{5}$$
- $$\text{and } a \rho_2 b \Leftrightarrow |a| = |b|, \forall a, b \in \mathbb{Z}$$
- Show that ρ_1 is a ring congruence on $\mathbb{Z}$ whereas ρ_2 is not. 3+2
14. (a) Let R be a ring and $Z(R) = \{x \in R : xr = rx, \forall r \in R\}$. Prove that $Z(R)$ be a subring of R .
 (b) Prove that the subring $S = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. 2+3
15. In a commutative ring R with unity, prove that an ideal I is prime if and only if the quotient ring R/I is an Integral Domain. 5
16. Let R be a commutative ring with unity and without divisors of zero. Prove that if R is finite then R is a field. 5
17. Let $(F, +, \cdot)$ be a field and $u(\neq 0) \in F$. Define multiplication $\times$ on F by
- $$a \times b = a \cdot u \cdot b, \forall a, b \in F.$$
- Prove that $(F, +, \times)$ is a field. 5

18. Let R be the ring $\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and $\phi : R \rightarrow \mathbb{Z}$ is defined by $\phi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b$. Show that ϕ is a ring homomorphism. Determine $\text{Ker } \phi$. Show that the ring $R/\text{Ker } \phi$ is isomorphic to $\mathbb{Z}$. 2+1+2
19. In a ring $(\mathbb{Z}_n, +, \cdot)$, show that an element $\bar{m}$ is a unit if and only if the $\gcd(m, n)$ is 1. 5
20. (a) Prove that the ring of matrices $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a field.
 (b) Prove or disprove that the ideal $4\mathbb{Z}$ is a prime ideal in a ring $2\mathbb{Z}$. 3+2
21. Show that the mapping $\phi : \mathbb{Z}[x] \rightarrow \mathbb{Z}$ defined by $\phi(f(x)) = f(0)$ for $f(x) \in \mathbb{Z}[x]$, the set of all polynomials in x over $\mathbb{Z}$, is a homomorphism. Find $\text{Ker } \phi$. 2+3
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