

2026

STATISTICS — HONOURS

Paper : DSCC-5

(Linear Algebra)

Full Marks : 75

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*

Answer **any five** questions out of **question nos. 1-8**, **any four** questions out of **nos. 9-14**
and **any three** questions out of **nos. 15-19**.

1. Find a third equation to $2x = 4y + 3z$ and $2x + y - z = 10$ such that $(x, y, z) = (5, 2, 2)$ is the unique solution to the system formed by the three equations. 2
2. If ' denotes the transpose and $\otimes$ denotes the Kronecker product of matrices, prove that for any two matrices A and B , $(A \otimes B)' = A' \otimes B'$. 2
3. If $A_{5 \times 5}$ is orthogonal and c a scalar such that $|cA| = 2^5$, what are the possible values of c ? 2
4. If A is a symmetric matrix, is the condition $r(A) = \text{tr}(A)$ (a) necessary, (b) sufficient for the idempotence of A ? Explain. 2
5. Give an example of two matrices A and B with AB defined, and $r(AB) < \min\{r(A), r(B)\}$. 2
6. If S and T are subspaces of a vector space and $S \subseteq T$, what is $S + T$? 2
7. Prove that the span of any set of vectors is always a subspace. 2
8. Prove that similar matrices have the same determinant. 2
9. If $\dim(V) \geq 3$ and $\underline{x}$ and $\underline{y}$ are linearly independent vectors in V , show that $\underline{x}$ and $\underline{x} + 3\underline{y}$ are linearly independent. Is it possible for $\underline{x}$, $\underline{x} + 2\underline{y}$ and $\underline{x} + 3\underline{y}$ to be also linearly independent? Justify your answer. 5
10. Which of the following are subspaces? Give reasons for your answer :
 - (a) The set of all solutions (x, y, z) to the equation $3x + 2y - z = 0$
 - (b) The set of all solutions (x, y, z) to the equation $3x + 2y - z = 4$
 - (c) The set of all solutions (x, y, z) to the equation $3x + 2y - z = 0$, where the components x, y and z are each non-negative. 5

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11. Find the dimension and a basis of the subspace $\{(x, y, z) : x + 2y + 3z = 0\}$ of $\mathbb{R}^3$. 5
12. Define column space of a matrix. If A and B are matrices so that AB is defined, show that the column space of AB is a subspace of the column space of A . State without proof a property of B that forces equality of the two for every A . 5
13. Show that a symmetric matrix defines an indefinite quadratic form if and only if it has both positive and negative eigenvalues. 5
14. State the Cauchy-Schwartz inequality and using it, prove that for a random variable taking values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ respectively, the variance is non-negative. 5
15. (a) Suppose x_1, x_2, x_3 are vectors with $x_1 \neq 0$. Show that $\{x_1, x_2, x_3\}$ is linearly dependent if and only if either x_2 is a scalar multiple of x_1 or x_3 is a linear combination of x_1 and x_2 . Can you generalise this to characterize linear dependence of a set $\{x_1, x_2, \dots, x_k\}$ of vectors?
- (b) If $x \in \mathbb{R}^n$ has norm 1, show that there exists an orthogonal matrix P whose first column is x .
- (c) Suppose $x = (2, -2, 0, -1)$ is a solution of the system of linear equations $Ax = b$. Give examples of vectors b_1 and b_2 such that the vector $(4, -4, 0, -2)$
- (i) is a solution of $Ax = b_1$,
- (ii) is not a solution of $Ax = b_2$.
- (d) Suppose $T: V \rightarrow V$ is defined by $T(x) = aT(x) + b$ where a is a scalar and b is a vector. Show that T is linear if and only if $b = 0$. 6+3+3+3
16. What is the dimension of the vector space $\mathcal{V}$ of all 2×2 matrices (over $\mathbb{R}$)? Let $\mathcal{U}$ denote the set of 2×2 upper triangular matrices. Prove that $\mathcal{U}$ is a subspace of $\mathcal{V}$. What is its dimension? If $\mathcal{T}$ denotes the set of 2×2 lower triangular matrices, show that $\mathcal{S} \cup \mathcal{T}$ is not a subspace. Identify $\mathcal{S} \cap \mathcal{T}$ and $\mathcal{S} + \mathcal{T}$. 2+4+2+3+4
17. A square matrix M is partitioned as $\left[\begin{array}{c|c} A_{m \times m} & B \\ \hline C & D_{n \times n} \end{array} \right]$ where A is nonsingular.
- (a) Let $F := D - CA^{-1}B$. Obtain matrices E and G such that $M \cdot \left[\begin{array}{c|c} I & E \\ \hline G & I \end{array} \right]$ is partitioned as $\left[\begin{array}{c|c} A & \dots \\ \hline \dots & F \end{array} \right]$.
- (b) Conclude that $|M| = |A| \cdot |F|$.
- (c) Conclude that M is nonsingular if and only if F is.
- (d) Suppose that $n = 1$, i.e. D (and hence F) is a scalar. Then prove that the $(m+n, m+n)$ th entry of M^{-1} is $1/F$. 6+2+3+4

18. (a) Prove or disprove : If B is a basis of a vector space V and S is a subspace of V , then B contains a basis of S .

- (b) When is a linear transformation said to be of full rank?

Suppose $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation and $n > m$. Show that the nullity of T is at least $n - m$.

Consider the statement (A) T has full rank; and (B) nullity of T equals $n - m$. Does (A) imply (B)? Does (B) imply (A)? 4+(3+3+5)

19. (a) Define (i) an orthogonal, (ii) an orthonormal set of vectors.

- (b) If $S = \{x_1, x_2, \dots, x_k\}$ is an orthonormal set of vectors and x is any vector, show that the orthogonal

projection of x on the span of S is $\sum_{j=1}^k a_j x_j$ where a_j is the inner product of x and x_j .

- (c) Show that an orthogonal set is always linearly independent. (3+2)+6+4