

2026

STATISTICS — HONOURS

Paper : DSCC-7

(Sampling Distributions and Statistical Inference-II)

Full Marks : 75

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*1. Answer **any five** questions :

2×5

(a) A random variable X has the distribution

$$P(X=0) = \frac{1}{2}, \quad P(X=1) = \frac{1}{2}.$$

If X_1 and X_2 are independent observations from this distribution, find the sampling distribution of the sample mean $\bar{X}$.

(b) Let $X_{(1)} < X_{(2)} < X_{(3)}$ be the order statistics of a random sample of size three from the standard Laplace distribution with probability density function (p.d.f.)

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$

Find $P(X_{(3)} > 0)$.

(c) Let $X_1, X_2, \dots, X_n$ be a random sample from

$$P(X=0) = 1-\theta, \quad P(X=1) = \theta, \quad 0 < \theta < 1.$$

Find a sufficient statistic for θ .

(d) If T is a random variable having a t -distribution with n degrees of freedom, identify the distribution of T^2 .

(e) Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a random sample from a bivariate normal population with means μ_X, μ_Y , variances σ_X^2, σ_Y^2 , and correlation ρ . State the joint distribution of $(\bar{X}, \bar{Y})$.

(f) How does a randomized test differ from a non-randomized test in terms of the test function?

(g) What is meant by a uniformly most powerful unbiased (UMPU) test?

(h) The observed value of the likelihood ratio statistic is close to 1. What does this indicate about the null hypothesis?

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2. Answer **any four** questions :

5×4

- (a) Let X_1 and X_2 be independent $N(\mu, \sigma^2)$. Define $\bar{X} = (X_1 + X_2)/2$ and $D = X_1 - X_2$. Show that $\bar{X}$ and D are independent.

- (b) Let X and Y be independent random variables, each having the Uniform (0, 1) distribution. Define

$$U = \min(X, Y), \quad V = |X - Y|.$$

Determine the marginal densities of U and V .

- (c) Explain the significance of Fisher's Information in statistical inference. How does it measure the amount of information about an unknown parameter contained in a sample?
- (d) Let $T(X)$ be an unbiased estimator of $g(\theta)$. Derive the necessary and sufficient condition for $T(X)$ to attain the Cramer-Rao lower bound.
- (e) Let X be a discrete random variable taking values $\{0, 1, 2, 3\}$. Its probability mass function depends on a parameter θ as follows :

x	0	1	2	3
$P_{\theta_0}(X = x)$	0.4	0.3	0.2	0.1
$P_{\theta_1}(X = x)$	0.1	0.2	0.3	0.4

Construct the most powerful test of size $\alpha = 0.3$ for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ using the Neyman-Pearson lemma. Clearly specify the rejection region.

- (f) Let $X_1, X_2, \dots, X_n$ be a random sample from Uniform (0, θ), $\theta > 0$, and let $X_{(n)}$ denote the sample maximum. Using the pivotal quantity $X_{(n)}/\theta$, construct the shortest $100(1 - \alpha)\%$ confidence interval for θ .

3. Answer **any three** questions :

- (a) (i) Let X and Y be two independent random variables. Suppose X has an exponential distribution with p.d.f.

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0,$$

and Y has an exponential distribution with p.d.f.

$$g(y) = \mu e^{-\mu y}, \quad y > 0, \mu > 0.$$

Define

$$U = X + Y, \quad V = \frac{X}{Y}.$$

Find the joint density of (U, V) and the marginal distributions of U and V . Are U and V independent?

- (ii) Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be the order statistics in a random sample of size n from the p.d.f.

$$f(x) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad x > \mu, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$

Derive the joint distribution of $X_{(1)} - \mu$ and $\sum_{i=1}^n (X_{(i)} - X_{(1)})$ and hence determine whether

they are independent.

8+7

- (b) (i) Let $X_1, X_2, \dots, X_{2n}$ be independent random variables distributed as $N(\mu, \sigma^2)$. Find the distribution of

$$T = \frac{1}{2\sigma^2} \sum_{i=1}^n (X_{2i-1} - X_{2i})^2.$$

Hence obtain $E(T)$ and $Var(T)$.

- (ii) Let $X_1, X_2, \dots, X_n$ be a random sample from

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{\theta})^2}, \quad -\infty < x < \infty, \quad \theta > 0.$$

Find a sufficient statistic for θ and obtain an unbiased estimator of θ based on the sufficient statistic.

8+7

- (c) Let $X_1, X_2, \dots, X_n$ be a random sample from a log-normal distribution with unknown parameters μ and σ^2 . Define

$$G = \left(\prod_{i=1}^n X_i \right)^{1/n}.$$

- (i) Derive the sampling distributions of G .

- (ii) Find the distribution of

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^n \ln X_i \quad \text{and} \quad \frac{(n-1)S_Y^2}{\sigma^2},$$

$$\text{where } S_Y^2 = \frac{1}{n-1} \sum_{i=1}^n (\ln X_i - \bar{Y})^2.$$

- (iii) Hence construct a $100(1 - \alpha)\%$ confidence interval for μ using a suitable pivot quantity and determine the length of the interval.

4+4+7

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(d) Let $(X_1, X_2, \dots, X_n)$ be a random sample from a Poisson distribution with parameter $\lambda(> 0)$. Define

$$Y = 1 \quad \text{if } X_1 = 0 \\ = 0 \quad \text{if } X_1 \neq 0.$$

(i) Find the function $g(\lambda)$ such that Y is an unbiased estimator of $g(\lambda)$.

(ii) Determine $\phi(T) = E(Y | T)$ where $T = \sum_{i=1}^n X_i$. Hence obtain an improved unbiased estimator of $g(\lambda)$ based on T . Explain briefly why it is an improvement over Y .

(iii) Using the statistic T , construct the likelihood ratio test of size α for

$$H_0 : g(\lambda) = g_0 \text{ against } H_1 : g(\lambda) = g_1, \quad g_1 < g_0. \quad 2+7+6$$

(e) Let $X_1, X_2, \dots, X_n$ be a random sample from $N(0, \sigma^2)$, $\sigma^2 > 0$ unknown.

(i) Obtain the Cramer-Rao lower bound for unbiased estimators of σ^2 . Hence identify an unbiased estimator of σ^2 which attains this lower bound.

(ii) Construct the uniformly most powerful test of size α for

$$H_0 : \sigma^2 = \sigma_0^2 \text{ against } H_1 : \sigma^2 > \sigma_0^2.$$

(iii) By inverting the test in (ii), obtain a $100(1 - \alpha)\%$ confidence set for σ^2 . Comment on whether it is uniformly most accurate. 5+6+4